A FIRST STEP IN EUCLID

CONSISTING OF THE

FIRST TWELVE PROPOSITIONS

WITH EXPLANATIONS, ILLUSTRATIONS, AND NUMEROUS EXAMPLES

BY

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A FIRST STEP IN EUCLID.

EUCLID
[Elementa. Appendix]

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PREFACE.

It is not here intended to discuss the advisability of teaching Euclidean Geometry to young boys. The fact is accepted that this most abstract subject forms part of the curriculum of our schools. The question arises, how far the study of it can be rendered intelligible, serviceable, and, if possible, agreeable to boys and girls.

Some teachers begin by instructing their pupils in the elements of geometrical drawing, a plan which is excellent in itself, but does not help them to overcome the initial difficulties of geometrical reasoning. It seems that two facts should be kept in view. Young minds can grasp practical demonstrations, whereas theoretical reasoning presents to them great difficulty. Again, so long as their natural enterprise is not stamped out by constant repression, but stimulated by suitable problems, they are generally ready to puzzle out questions for themselves. Therefore, in the first place, Euclid should be introduced to the beginner with the help, as far as is possible, of practical illustrations. Take a class out of doors, and illustrate the application of I. 4 to indirect measurements; their interest will be aroused, and in future there will be little of the bewildering muddle-headedness connected with that proposition, of which the majority of teachers of a low class in Euclid are so painfully aware. And, in the second place, let the pupils be encouraged to build up for themselves their own geometry. An attempt has been made, though a very imperfect one, in this little book to indicate how the pupil, in certain propositions, may deduce the proof for himself. But this must rest rather in the hands of the teacher than of the compiler.

Orthodox geometricians will find here much scope for criticism, but then it does not often fall to the lot of our orthodox geometricians to teach the elements of the subject.

Those who adopt this book will probably prefer their own way of using it, but it is well to state the plan of teaching the subject that prompted the writing of it. It is intended that no propositions should be learnt and written out, until the geometrical facts and proofs contained in the first twelve propositions have been understood through oral teaching. The questions on the axioms are designed for oral teaching, while the questions on page 12 give practice in handling compasses and ruler; but it is recommended that at this stage no written explanation of their drawings should be exacted from the beginners. Let the first attempt to write out any geometrical reasoning be made with the riders on pages 32 and 33. But even here it is best to go through the greater part of the exercise *vivâ voce*, before the class is allowed to commit anything to paper. After they have acquired some skill in doing these very easy riders, an attempt may be made to learn the propositions with a view to writing them out, but it is generally best to postpone this as long as possible, for the pupil gains in power by depending on his own reasoning, instead of being set to learn the deductions made by others. At any rate his first trial should be with the shortest and not the longest propositions.

The purely Euclidean part of the book has been reprinted at the end in a consecutive form, partly for reference, and partly to help the beginner to distinguish it from the explanatory matter.

I have to thank my former colleagues at Clifton, Mr. H. S. Hall, and Mr. H. G. Barlow, for help and suggestions while the book was passing through the press, and Mr. H. S. Hall, and Mr. F. H. Stevens, for permission to borrow from their unrivalled edition of Euclid, to which this is intended as an easy introduction.

PACKWOOD HAUGH, HOCKLEY HEATH,
November, 1894.

A FIRST STEP IN EUCLID.

CHAPTER I.

DEFINITIONS.*

1. A **point** is that which has position, but no magnitude.

2. A **line** is that which has length without breadth.

The extremities of a line are points, and the intersection of two lines is a point.

3. A **straight line** is that which lies evenly between its extreme points.

4. A **surface** is that which has length and breadth, but no thickness.

The boundaries of a surface are lines.

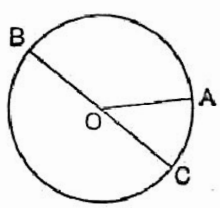
5. A **plane surface** is one in which any two points being taken, the straight line between them lies wholly on that surface.

A plane surface is frequently referred to simply as a plane.

10. Any portion of a plane surface bounded by one or more lines, straight or curved, is called a **plane figure**.

* A few definitions only are given here. See page 63 for the complete list.

11. A **circle** is a plane figure contained by one curved line, which is called the **circumference**, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another: this point is called the **centre** of the circle.



A **radius** of a circle is a straight line drawn from the centre to the circumference.

12. A **diameter** of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

It is necessary to understand clearly what Euclid means by a *point* and a *line.*

A point is considered as a mere mark of position and is without any area of its own.

A line has no breadth: it is an indefinitely thin boundary.

Now it is impossible to represent such a point or such a line on paper, for they ought not to be visible. When a rough sketch is made of a house or tree, lines more or less thick are drawn to denote the boundary of the brick or wood. But there are no such thick lines in nature. So in the same way we have to draw lines more or less thick to denote Euclid's lines, but we must remember that mathematical lines have really no depth or breadth.

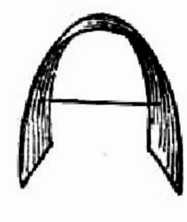
A *surface* must not be confused with a *solid body.* It is merely the outside of a solid body, and cannot even be called a film or shell, for then it would have some depth, however small.

Many surfaces are smooth, but not *plane.*

The surface of a new garden roller is smooth. But take several points on the roller and join them by a straight line, and the joining lines will not always lie along the surface.

Again, the lens of a microscope or telescope is smooth, but not plane.

Mark several points on a sheet of paper. If your sheet of paper is lying down flat, the straight lines which join them lie on the surface of the paper. So now your sheet of paper is a plane surface. But if you make an arch of the paper, as in the accompanying figure, then the straight lines, which join some of the points, lie outside of the surface of the paper. And so your sheet of paper is no longer a plane surface. It is possible to take two points such that the straight line joining them does lie on the surface. But those are particular points, and the definition implies that whatever points are taken, the line joining them must lie on the surface.



Now if on our flat sheet of paper which represents a plane surface we draw a line or any number of lines enclosing a space, the spaces so enclosed are called *plane figures.* They are called "plane" because the figure is drawn on a plane surface. If the sheet of paper is crumpled or curved into an arch, the spaces enclosed are still figures, but no longer plane figures.

Now on every printed page there are very many figures formed by the printed letters. Some letters, such as an 'r,' enclose no figure; others, such as 'o,' enclose one figure; while there are two letters each of which enclose two figures, viz., 'B' and 'g.' According to their shape, these plane figures are called by various names. At present we are only dealing with one, viz., the *circle.* We notice that it is bounded by a single curved line (not a *straight* line, as many blunderers say). To draw it, we use a pair of compasses. We place the sharp point of the compasses at O (see figure on page 2), the centre of the circle; we extend our other leg till it stands at A: finally we revolve our compasses, keeping the fixed point at O, and moving the other leg round. This other leg will mark out the *boundary* of our figure. The boundary of the circle is called its *circumference.* Where there is no fear of confusion the word 'circle' is often used to signify the circumference. But properly, a circle is a 'figure,' and the circumference a 'boundary.'

The distance OA to which we have stretched our compasses is called a *radius* of the circle. The line BOC is a *diameter*. OB and OC are radii. So we see that a diameter is double of a radius.

POSTULATES.

LET IT BE GRANTED:

1. That a straight line may be drawn from any one point to any other point.

2. That a finite, that is to say, a terminated straight line may be produced to any length in that straight line.

3. That a circle may be described from any centre, at any distance from that centre; that is, with a radius equal to any finite straight line.

These *postulates* are demands for the use of a ruler and a pair of compasses.

The first two postulates demand that a ruler should be allowed for joining points or producing straight lines.

The third postulate demands that a pair of compasses should be used to draw a circle from any given point with any radius already given in position.

Thus it will be seen that, *for the purpose of measuring*, Euclid does not sanction the use of a ruler, or of compasses. We are not allowed to *carry* our compasses for marking off equal distances.

It may be objected that this is rather an absurd restriction. In his second proposition he shows how a line may be drawn from any given point equal to any given straight line, and the proposition is interesting as a piece of ingenuity. As a matter of fact, nobody who wished to draw a line equal to another line, say an inch long, would hesitate to carry his compasses, and in the second chapter we have allowed the student to carry his compasses.

AXIOMS.

An **axiom** is that which is evident to the senses, and which cannot be made clearer by proof.

GENERAL AXIOMS.*

1. Things which are equal to the same thing are equal to one another.

2. If equals be added to equals, the wholes are equal.

3. If equals be taken from equals, the remainders are equal.

4. If equals be added to unequals, the wholes are unequal, the greater sum being that which includes the greater of the unequals. cf 14?

5. If equals be taken from unequals, the remainders are unequal, the greater remainder being that which is left from the greater of the unequals. cf 14 ?

6. Things which are double of the same thing, or of equal things, are equal to one another.

7. Things which are halves of the same thing, or of equal things, are equal to one another.

9. The whole is greater than its part.

It is not likely that any one will be found who will object to the self-evident nature of the eight axioms given above.

It more frequently happens that the beginner is not only willing to accept the truth of the axioms, but considers that

* For the axioms in full, see page 66.

there are many other things in geometry which might be accepted without proof. If so, he may find himself accepting statements to be universally true which are in reality only true in particular cases.

Illustration of Axiom 1.*

Two guards, one at Windsor, the other at Buckingham Palace, had a dispute by letter about their comparative heights. Each claimed to be the taller. But they were unable to leave their posts. So a comrade of equal height to the Windsor guard was sent up to London, and was found to be equal in height also to the guard at Buckingham Palace. The two guards were thus each equal in height to the "*traveller*," and even to their limited intelligence it was obvious that they must be equal in height to one another.

In the language of Euclid, *things which are equal to the same thing are equal to one another.* The two guards were "*the things*," and the traveller "*the same thing.*"

Exercise.

In the following examples state the conclusion and point out the "traveller."

(1) Tom and John are both the same age as I, therefore . . .

(2) Volumes II. and III. are the same length as Vol. I., therefore . . .

(3) I have the same amount of money as Brown or Smith, therefore . . .

(4) $A = B$ and $C = B$, therefore . . .

Which of the axioms are illustrated by the following statements?

(1) Brown has as much money as Smith, and Jones as Robinson, therefore Brown and Jones together have as much money as Smith and Robinson.

(2) Two armies, equal in number, each lose 500 men in a battle, consequently they still have the same number of men.

* This illustration is due, I believe, to the late Mr. Hawtrey.

(3) Brown and Smith are each double the height of the dwarf Jones, therefore they are the same height as one another.

(4) My whole hand is larger than my thumb.

(5) Tom has more money than John. They each lose a shilling, therefore still has Tom more money than John.

(6) I am older than you. In five years' time I shall still be older than you.

(7) Smith and Brown are each the same age as Jones, therefore they are the same age as one another.

(8) One half of a book is as long as the other half.

Geometrical Exercise on the Axioms.

N.B.—The sides of a *square* are all equal.

1. In Figure 2* prove that FC is equal to OD.

2. In Figure 1 prove that (1) AK = AN; (2) KE = NF; (3) EB = FD; (4) AE is greater than AN; (5) AF is less than AB; (6) the sum of AB, BL = the sum of DC, CL.

3. In Figure 5 prove that EA is equal to AG.

4. In Figure 3 prove that (1) EB = DG; (2) AD is greater than EA; (3) the sum of AB, AG = the sum of AD, AE.

5. In Figure 4 prove that (1) AC = BD; (2) OR is greater than OQ; (3) AR = QD.

6. In the figure of Example 1 on page 11 ABCD is a square, and with centres A and C two circles with equal radii have been described. Prove that BM and PB are equal, and also DN and BM.

7. In the figure on page 16 DE, DF are radii of the larger circle and BE, BC of the smaller circle, and DBA is a triangle with all its sides equal. Prove that AF = BE, and hence that AF = BC.

The following examples on the axioms may be tried after Chapter IV. has been read:

8. In Figure 5 prove that the angles EAG, FBC are equal, and that the angle EAB is greater than FBA.

* Figures 1, 2, 3, 4, 5 will be found on pages 83, 85.

9. In the figure on page 34, if it is given that the angles ABG, ACF are equal, and also the angles CBG, BCF are equal, prove that the angles ABC, ACB are equal.

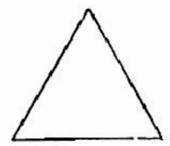
10. In the figure on page 44, if it is given that the angles ACD, ADC are equal, prove that (1) the angle ADC is greater than the angle BCD; (2) the angle BCD is greater than the angle ACD.

CHAPTER II.

FURTHER DEFINITIONS.

16. A **triangle** is a plane figure bounded by three straight lines.

19. An **equilateral** triangle is a triangle whose three sides are equal.



20. An **isosceles** triangle is a triangle, two of whose sides are equal.

Any one of the angular points of a triangle may be regarded as its **vertex**; and the opposite side is then called the **base.** In the case of an isosceles triangle the term base is only applied to the side which is not equal to the other two, and the term vertex to the angle opposite to that side.

It has been already remarked that Euclid apparently considered that distances could not be accurately measured by *carrying* the compasses. In the next chapter we shall show how he gets over this difficulty. Meantime, in the present chapter we shall assume that we are able to measure distances.

DRAWING TO SCALE.

In drawing maps or plans it is necessary to represent actual distances by very much shorter lines. If we want to represent a large tract of country in a small map, an inch on the map may represent 10 miles, or even 50 or 100 miles. If we wish to have a fairly detailed map, we represent only 1 mile of the district by an inch. If we are drawing a plan of a house, probably each inch of the plan is made to denote only a few feet or yards. If our scale is an inch to the mile, then 2 inches denote 2 miles, $\frac{1}{2}$ inch denotes $\frac{1}{2}$ mile, 1 yard denotes 36 miles, and so on.

Exercise.

(1) On the scale of an inch to the mile draw lines to represent 3 miles, $4\frac{1}{2}$ miles, $\frac{3}{4}$ mile, $2\frac{1}{2}$ miles, 1 mile, 3 furlongs.

(2) On the scale of $\frac{1}{2}$ inch to the mile draw lines to represent 4 miles, 5 miles, $3\frac{1}{2}$ miles, $2\frac{1}{4}$ miles.

(3) On the scale of $\frac{1}{8}$th of an inch to the yard, draw lines to represent 7 yards, 12 yards, 24 yards, 29 yards.

ON CIRCLES.

Before drawing a circle it is necessary to know two things:

(1) The point where the sharpened leg of the compass must be placed, *i.e.* the position of the centre.

(2) The extent to which the compass must be stretched, *i.e.* the length of the radius.

It will be seen that by means of the circle we can find the position of points, all of which lie at the same distance from some fixed point. For instance, all points one mile away from a given point A lie on a circle,* whose centre is A and radius equal to one mile.

In practical life we may represent by a circle the boundary of the ground that may be grazed by a tethered animal, or reached by a garden syringe, or commanded by a gun, etc., etc. To enable us to draw a plan of such boundaries we must know

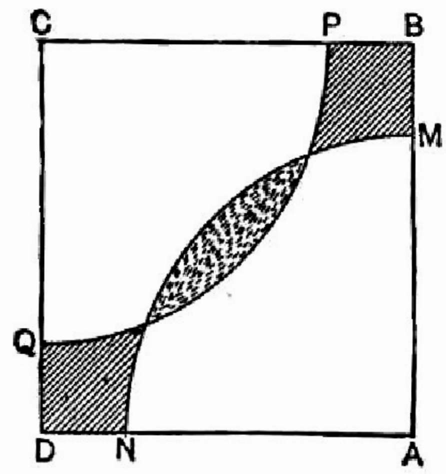
(1) The position of the fixed point, *i.e.* the centre of our circle.

(2) The length of the range of the tethered animal, syringe, or gun, *i.e.* the radius of our circle.

* A "circle" here means "the circumference." See page 4.

Example I.

Let ABCD represent a small square plot of grass, each of whose sides is 20 yards. Cows are tethered to the opposite corners A and C with ropes 15 yards long. Draw a figure to show which part of the grass cannot be eaten, and which part can be eaten by both. (Scale 20 yards to the inch.)



15 yards will be represented by $\frac{15}{20}$ of an inch, *i.e.* $\frac{3}{4}$ inch.

The cow tethered at A can graze for a distance of 15 yards from A.

Hence it can graze over the ground enclosed in a circle whose centre is A and radius 15 yards.

Therefore with centre A and radius $\frac{3}{4}$ inch, describe a circle cutting AB, AD in M and N.

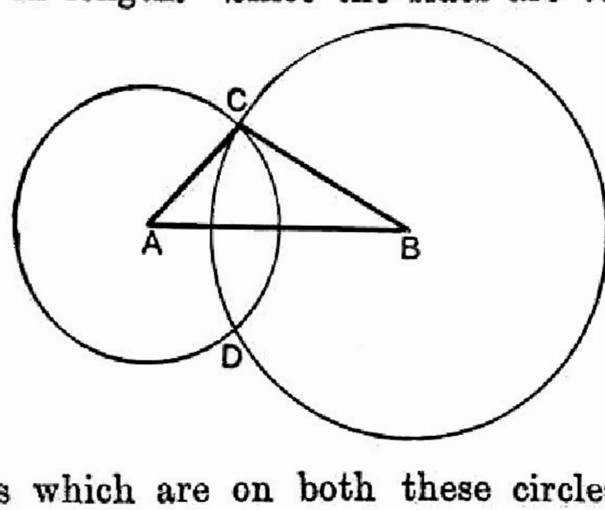
Similarly, with centre C and radius $\frac{3}{4}$ inch, describe another circle, cutting CB in P and CD in Q.

Then the shaded part cannot be touched by either cow, the dotted part can be eaten by both.

Example II.

On a straight line $1\frac{1}{16}$ inch in length as base draw a triangle having its other two sides equal to $\frac{1}{3}$ inch and $\frac{1}{2}$ inch.

Let AB be a straight line $1\frac{1}{16}$ inch in length. Since the sides are to measure $\frac{1}{3}$ inch and $\frac{1}{2}$ inch, it follows that the third angular point must be $\frac{1}{3}$ inch from A, and $\frac{1}{2}$ inch from B. Since it is to be $\frac{1}{3}$ inch from A, it must lie on the circumference of a circle, whose centre is A and radius $\frac{1}{3}$ inch. Since it is to be $\frac{1}{2}$ inch from B, it must lie on the circumference of a circle, whose centre is B and radius $\frac{1}{2}$ inch. Now the only points which are on both these circles are the points where the circles intersect.



If these points are C and D, then the triangle ACB has its sides of the required length. Also if A and B are joined to D, the triangle ADB will have its sides of the required length.

Exercise.

1. Draw a figure to show the amount of ground that can be covered by a fixed syringe, which can squirt water a distance of 24 yards. (Scale $\frac{1}{16}$ inch to the yard.)

2. A man stands on the edge of a circular grass plot of radius 16 yards, and waters it with a syringe that carries 24 yards. Draw a figure to show how much will be watered. (Scale $\frac{1}{16}$ inch to the yard.)

3. A donkey is tethered 8 yards from a long straight hedge, and can graze a distance of 12 yards from its tether. Draw a figure to show how much of the hedge it can nibble. (Scale $\frac{1}{16}$ inch to the yard.)

4. In question 2, another man with an equally powerful syringe stands on the circumference at the farthest point reached by the first man. Show what part of the ground will be watered by both.

5. Two forts containing guns that can carry 2 miles are 3 miles apart. Draw a figure to show what space will be under fire from both forts. ($\frac{1}{2}$ inch to the mile.)

6. A and B are any two points 3 inches apart. Draw a figure upon which all points 2 inches distant from A must lie. Draw another figure upon which all points 2 inches distant from B must lie. How many points are there distant 2 inches both from A and B?

7. A and B are any two points 1 inch apart. CBD is any straight line passing through B, but not through A. Draw a figure on which all points $1\frac{1}{2}$ inches from A lie, and find two points in CBD each $1\frac{1}{2}$ inches from A.

8. York and Harrogate are 18 miles apart in direct line. Find two points which are each 8 miles from York and 16 miles from Harrogate. (Scale $\frac{1}{16}$ inch to the mile.)

9. AB is any straight line of limited length, and DAC another line of unlimited length passing through A. Find another point besides A in DAC, which is the same distance from B as A is from B.

10. AB is a line 1 inch long. Draw a circle on which will lie all points 2 inches distant from A, and another on which will lie all points 2 inches distant from B Hence draw two isosceles triangles on the base AB, with their sides 2 inches long.

11. M and N are two towns 4 miles apart. A straight road passes through N. Find two points on it, each 5 miles from M. (Scale $\frac{1}{4}$ inch to the mile.)

12. A and B are two points 1 inch apart. CBD is any line passing through B. Draw an isosceles triangle with vertex A and base on CBD having sides 2 inches long.

13. A and B are points 2 inches apart. Draw a circle on which will lie all points 2 inches distant from A, and another on which will lie all points 2 inches distant from B, and let one of the points in which the circles cut be C. If A, B, and C are joined, what kind of triangle is formed?

Anyone who has been able to do this last question will have little difficulty in drawing an equilateral triangle on any given base AB, whatever may be the distance from A to B. For we have the position of two angular points of our triangle, viz., A and B, and only need to find the position of the third, which we will call C.

Now C has to be the same distance from A, as B is from A. Therefore it must lie on the circumference of a circle whose centre is A and radius AB.

Again C has to be the same distance from B, as A is from B. Therefore it must lie on the circumference of a circle whose centre is B and radius BA.

These circles only cut in two points, and so C must be one of these points.

It is not unlikely that by some such process of reasoning as the above, Euclid or his predecessors arrived at the construction of his first **proposition.**

He writes out his propositions in a very clear way, giving the reason for each step, and the student in course of time has to learn to write them out in a similar way.

Proposition is the name given to each separate discussion. Propositions are of two kinds, **Problems** and **Theorems.**

A **problem** proposes to effect some geometrical construction: *e.g.* in the first proposition it is proposed *to describe an equilateral triangle on a given finite straight line.*

This is called the **general enunciation** of the proposition.

A problem consists of four parts: I., general enunciation; II., particular enunciation; III., construction; IV., proof.

I. The general enunciation names

(1) the **data**, or things given, *e.g.* a finite straight line.

(2) the **quaesita**, or things required, *e.g.* to describe an equilateral triangle on it.

II. The **particular enunciation** repeats the substance of the general enunciation, but refers to a special figure marked with letters. It names

(1) the **data**, *e.g.* let AB be the given straight line.

(2) the **quaesita**, *e.g.* it is required to describe an equilateral triangle on AB.

III. The **construction** directs the drawing of such lines or figures as may be necessary to effect the purpose of the problem, *e.g.* in Proposition I. the necessary circles are directed to be drawn.

IV. The **proof** demonstrates that the object of the problem has been effected, *e.g.* in Proposition I. the triangle ABC is proved to be an equilateral triangle.

A **theorem** proposes to demonstrate some geometrical truth, *e.g.* if two circles cut one another, they cannot have the same centre.

In a theorem the enunciation consists of

(1) the **hypothesis**, or that which is assumed or supposed to exist, *e.g.* (in the above) it is supposed that two circles cut one another.

(2) the **conclusion**, or the assertion to be proved, *e.g.* (in the above) it is to be proved that the circles cannot have the same centre.

In other respects the parts of a theorem are similar to those of a problem.

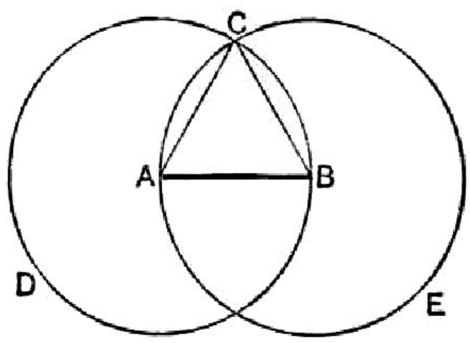
At the end of a **problem** there are written the letters Q.E.F., which are the initial letters of **quod erat faciendum**, "which was to be done."

At the end of a **theorem** the letters Q.E.D. stand for **quod erat demonstrandum**, "which was to be proved."

PROPOSITION 1. PROBLEM.

General Enunciation.

To describe an equilateral triangle on a given finite straight line.



Particular Enunciation.

Let **AB** be the given finite straight line.
It is required to describe an equilateral triangle on **AB**.

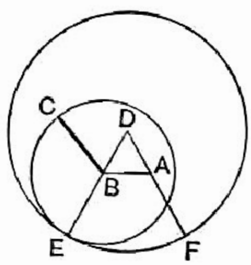
Construction.

From centre **A** with radius **AB**, describe the circle **BCD**.
From centre **B** with radius **BA**, describe the circle **ACE**.
Let the circles intersect in **C**.
Join **CA** and **CB**.
Then shall **ABC** be an equilateral triangle.

Proof. Because **A** is the centre of the circle **BCD**,
therefore **AC = AB**,
and because **B** is the centre of the circle **ACE**,
therefore **BC = BA**.
Therefore **AC** and **BC** are each equal to **AB**.
But things which are equal to the same thing are equal to one another. Ax. 1.
Therefore **AC = BC**.
Therefore **CA, AB, BC** are equal to one another.
Therefore the triangle **ABC** is equilateral;
and it is described on the given straight line **AB**. Q.E.F.

CHAPTER III.

WE will now show how Euclid draws a straight line from a given point equal to a given straight line *without* carrying the compasses.



Let A be the given point, BC the given straight line.

He joins A to one extremity of the straight line, say to B.

With centre B and radius BC he describes a circle.

On AB, *on either side of it*, he describes an equilateral triangle BAD, and produces DB to meet the circle just drawn in E.

With centre D and radius DE he describes another circle, and produces DA to cut it in F.

Then he proves that AF is equal to BC.

[NOTE. In this construction be careful to produce DB, *not* DA or AB, to meet the small circle. A little thought will convince any one of the reason for this step.]

Now to prove that AF is equal to BC, we must use a "traveller," and here our "traveller" is BE.

It is quite obvious that BC is equal to BE, for they are radii of the same circle.

By Axiom 3 we can also prove AF equal to BE.

For DF and DE are a large pair of equals,

and DA and DB are a small pair of equals,

and if we take the small pair from the large pair, each from each, we have the remainders AF and BE.

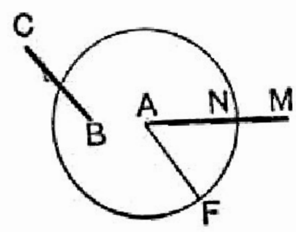
Hence we see that AF must be equal to BE.

Therefore "our two guards" AF and BC are each equal to the "traveller" BE, and so they are equal to one another.

It is a good thing to practise drawing this figure with A and BC in many different positions. If care is taken to produce DB, and not any other line, to the circumference of the small circle, little difficulty should be found in doing this. In other words, join the vertex of the equilateral triangle described on the base AB to B, and produce that line onwards beyond B to the circumference of the circle described about B as centre.

Now suppose in the above figure there was any straight line AM drawn from A to any distance. If we were asked to cut off from AM a part equal to BC, we should with centre A and radius AF draw a circle cutting AM in N. Then AN can be proved equal to BC.

In the accompanying figure the lines BC, AF only of the previous figure have been reproduced.



Here AF is our "traveller."

AN is equal to AF, for they are radii of the same circle.

BC has already been proved equal to AF.

Therefore AN and BC, being both equal to AF, are equal to one another.

This is the way Euclid cuts off a part of a straight line equal to another straight line.

But he does not go through all the construction.

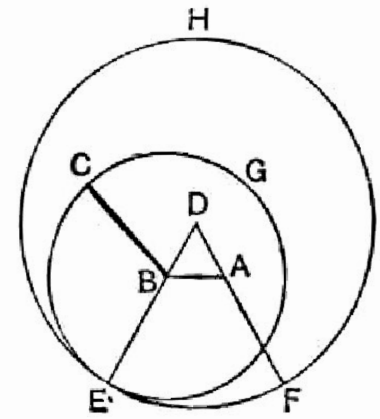
He wants to cut off from AM a part equal to BC.

He has already shown how a line AF *can* be drawn from A equal to BC. So he merely says, "From the point A draw the straight line AF equal to BC." And this is his way throughout. When he has *once* shown how to draw an equilateral triangle, or how to find the middle point of a straight line, or how to draw a perpendicular, he never repeats the construction.

PROPOSITION 2. PROBLEM.

General Enunciation.

From a given point to draw a straight line equal to a given straight line.



Particular Enunciation.

Let **A** be the given point, and **BC** the given straight line.

It is required to draw from the point **A** a straight line equal to **BC**.

Construction.

Join **AB**,

and on **AB** describe an equilateral triangle **DAB**. I. 1.

From centre **B**, with radius **BC**, describe the circle **CEG**.

Produce **DB** to meet the circle **CEG** at **E**.

From centre **D**, with radius **DE**, describe the circle **EFH**.

Produce **DA** to meet the circle **EFH** at **F**.

Then **AF** shall be equal to **BC**.

Proof.

Because **B** is the centre of the circle **CEG**,

therefore **BC** = **BE**,

and because **D** is the centre of the circle **EFH**,

therefore **DF** = **DE**,

and **DA**, **DB**, parts of them, are equal.

Therefore the remainder **AF** = the remainder **BE**. Ax. 3.

And it has been shown that **BC** is equal to **BE** ;

Therefore **AF** and **BC** are each equal to **BE**.

But things which are equal to the same thing are equal to one another. Ax. 1.

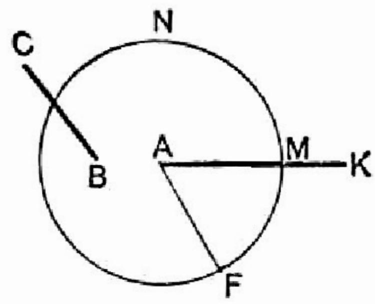
Therefore **AF** = **BC**,

and it has been drawn from the given point **A**. Q.E.F.

PROPOSITION 3. PROBLEM.

General Enunciation.

From the greater of two given straight lines, to cut off a part equal to the less.



Particular Enunciation.

Let **AK** and **BC** be the two given straight lines, of which **AK** is the greater.

It is required to cut off from **AK** a part equal to **BC**.

Construction.

From the point **A** draw the straight line **AF** equal to **BC**, I. 2.
and from centre **A**, with radius **AF**, describe the circle **FMN** meeting **AK** at **M**.

Then **AM** shall be equal to **BC**.

Proof.

Because **A** is the centre of the circle **FMN**,
therefore **AM** = **AF**.
But **BC** = **AF**,
therefore **AM** and **BC** are each equal to **AF**,
therefore **AM** = **BC**, Ax. 1.
and it has been cut off from the given straight line **AK**. Q.E.F.

CHAPTER IV.

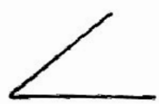
FURTHER DEFINITIONS.

6. A **plane angle** is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.

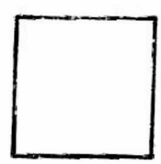
7. When a straight line, standing on another straight line, makes the adjacent angles equal to one another, each of the angles is called a **right angle**; and the straight line which stands on the other is called a **perpendicular** to it.

8. An **obtuse** angle is an angle which is greater than a right angle.

9. An **acute** angle is an angle which is less than a right angle.



28. A **square** is a four-sided figure which has all its sides equal, and all its angles right angles.



GEOMETRICAL AXIOMS.

8. Magnitudes which can be made to coincide with one another are equal.

10. Two straight lines cannot enclose a space.

11. All right angles are equal.

If a half-crown is taken up and laid upon another half-crown so that the edges of the first fall exactly over the edges of the second, the under surface of the upper coin is said to *coincide* with the upper surface of the lower coin.

If a piece of paper is laid upon another, so that each edge of one lies exactly over each corresponding edge of the other, the surfaces of the pieces of paper are said to *coincide*.

This method is called **superposition**, because one magnitude is *placed over* the second magnitude.

With two straight pencils try to enclose a marble. The truth of Axiom 10 will then be soon admitted.

ON ANGLES.

An **angle** is *not* an area.

The shape of a field may be a triangle, a square, or a circle, but it cannot be an angle.

Two lines make an angle with one another when they are different in direction and the angle measures the amount of this difference in direction, and the magnitude of the angle is not altered by the length of the lines.

If two men A and B start from the same point, A walking in a northerly and B in a north-easterly direction, the directions of their routes make a certain angle, and that angle is not changed whether they walk 10 yards or 10 miles. If a third man C walk only one yard in an easterly direction, his route would have made a greater angle with A's direction than B's route had done, even though B had walked 10 miles.

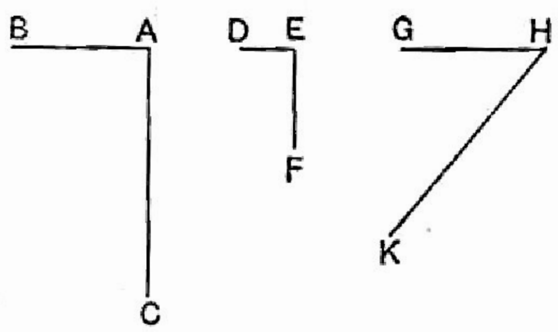
At any time of day, say at one o'clock, the hands of all correct clocks make the same angle with one another. The hands of Big Ben make the same angle as the hands of the tiniest bijou watch.

In the figure on page 51, the lines ED, DF are shorter than the lines CA, AF, but ED, DF contain a larger angle than CA, AF.

You may be directed along a road and told to turn to the right at a big angle ("abruptly" would probably be the word used), whether after turning to the right you walked straight on for 100 yards or 5 miles, you would have turned through the same angle.

The lines which meet and form an angle are said to *contain* that angle.

But equal angles may be contained by lines of unequal length, *e.g.* BA and AC are greater than DE, EF, but contain an equal angle (see accompanying figure); and if there are two pairs of lines of equal length, these pairs do not necessarily contain equal angles, *e.g.* the pair of lines BA, AC are equal to the pair GH, HK, but the angles they contain are not equal.



Again DE, EF are shorter than GH, HK, but contain a larger angle.

The point where the arms of the angle meet is called the **vertex** or **angular point**. The angle contained by BA, AC is called the angle BAC or the angle CAB, or sometimes the angle A, when there is only one angle at A.

Mention the angle contained by the lines :

(1) EB and BL in Figure 1.
(2) OB and BD in Figure 2.
(3) DO and EO in Figure 2.
(4) QO and OA in Figure 4.
(5) FB and BC in Figure 5.
(6) BC and FC in figure on p. 34, Prop. 5.

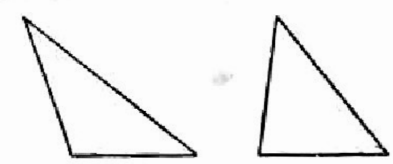
Mention the lines which contain the angles :

(1) ADC, DCB in Figure 1.
(2) OEB, ODE, OBE in Figure 2.
(3) POB and AOB in Figure 4.
(4) GAE and GFB in Figure 5.

ON TRIANGLES.

Triangles have six parts, viz. three angles and three sides. Two triangles are said to be *identically equal* or *equal in all respects* when one can be placed over the other so that it exactly covers it. Then all the six parts of the one will be equal to the six parts of the other.

Triangles may be of the same *size* though different in *shape*. This is expressed by saying that the triangles are equal in *area*. The triangles in the accompanying figure are equal in *area*, but are not *identically equal*.



Now, if it is known that three of the parts of one triangle are equal to three of the parts of another triangle, it can *often* be proved that the remaining parts are equal.

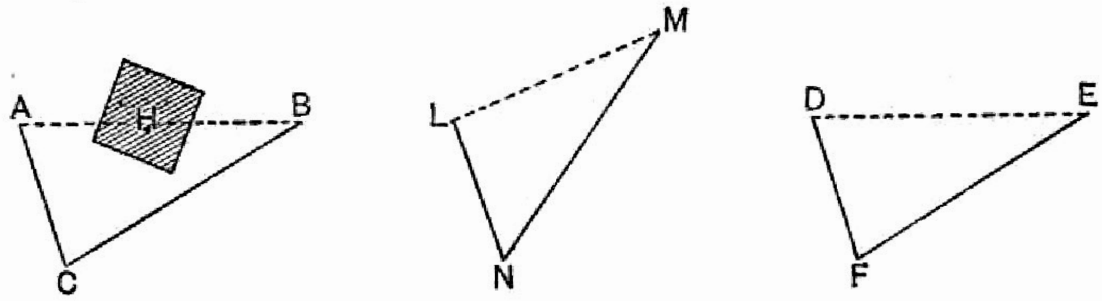
But this is not always the case. For instance, if we are told that the three angles of one triangle are equal to the three angles of the other triangle, it does not follow that the three sides are equal. For we can have two triangles, one very small, the other very large, which have all their angles equal; *e.g.* in Figure 1, if AC and NM are joined, the triangle ACD has its angles equal to those of the triangle DNM.

INDIRECT MEASUREMENT.

Now it is of the greatest importance in geometry to be able to prove that triangles are equal in *all* their parts when we know that *certain* parts are equal.

Geometry means the science of measuring the earth, and deals to a great extent with the measurement of distances that cannot be measured *directly*. We can measure directly the height of a man or length of a road, but a yard-measure alone is of little help for discovering the distance of a star, or even the height of a mountain. Even the following problem might prove difficult.

Suppose that A *and* B *are two stakes with some impenetrable body* H *(say a haystack) interposed between them. Find the distance from* A *to* B.



It is agreed that some *indirect* method must be employed, by drawing lines in some other part of the field, where there are no obstacles in the way.

The point C is taken in such a position that its distances from A and B can be measured directly.

Mr. Simpleton takes another point N, draws a line NL equal to CA, and another NM equal to CB. He joins LM, and measures it, and declares AB must be equal to LM.

Mr. Wiseman, however, takes a point F, and draws FD, FE equal to CA and CB, but *also* makes the angle DFE equal to the angle ACB. He measures DE and declares that AB must be the same length as DE. Which of the two is right? Can Euclid tell us?

You will notice that the two triangles ABC, DEF have three parts equal: viz. the two sides AC, CB equal to the sides DF, FE, *and also* the angle ACB, contained by AC, CB, equal to the angle DFE, contained by DF, FE.

Does it follow that the side AB is equal to the side DE?

We must pause to consider this.

THE EQUALITY OF TRIANGLES.

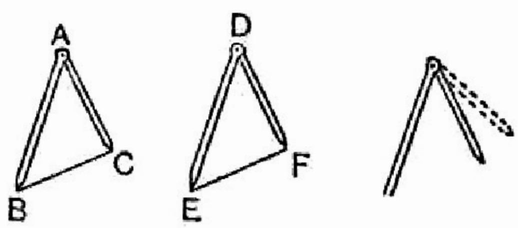
I can take up a pencil and lay the top of it exactly over the top of another pencil, and so that its length lies along the pencil underneath.

Will the point of the one lie exactly over the point of the other?

Not unless the pencils are of equal length.

Suppose I have two pairs of exactly similar compasses, and place the pivot of one compass exactly over the pivot of the other compass, and one leg of the first over one leg of the second, will the other leg of the first compass lie exactly over the other leg of the second compass? Not unless the compasses are equally opened, or in other words not unless the angles contained by the legs of the compasses are equal.

Now let AB, AC be two rods* jointed together at A, and DE, DF another pair jointed together at D, and let AB be equal to DE, and AC to DF, and also let the angle BAC be equal to the angle EDF.



Then we can place the first pair down on the second pair so that A lies on D and AB along DE.

Then B will fall on E, because AB is equal to DE.

And AC will lie along DF because the angle BAC is equal to the angle EDF.

If these angles were unequal the rod AC might fall as the dotted lines in Figure 3.

And because AC lies along DF, and is equal to it in length, the point C will fall on F.

If AC was not equal to DF, AC would either not completely cover or would overlap DF.

If now BC and EF are joined by straight strings, the string BC must lie along EF.

For the end B falls on E, and the end C on F, and if BC does not lie upon EF they will enclose a space, which cannot be. (See Axiom 10.)

So BC falls exactly on EF, and is equal to it.

Also the whole triangle ABC falls exactly on DEF and must be equal to it.

And the angle ABC falls exactly on the angle DEF, and must be equal to it, and similarly the angle ACB must be equal to the angle DFE.

* Rods jointed together by a pivot are here used, as they can be easily made and employed to illustrate to a class the method of "superposition."

Hence we see that if two triangles have

(1) two sides of the one equal to two sides of the other, each to each, *and*

(2) the angle contained by the two equal sides of the one equal to the angle contained by the two equal sides of the other,

we draw three *conclusions*, viz. that

(1) the third sides are equal;

(2) the triangles are equal in area;

(3) the remaining angles of the one are respectively equal to the remaining angles of the other.

And so it is evident that Mr. Wiseman was right, and that it is necessary that the *contained angles also shall be equal.*

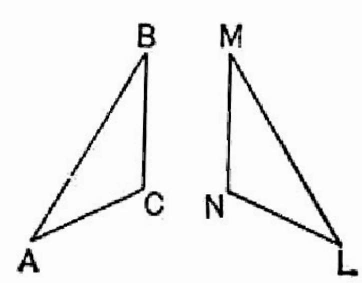
But driven into a corner Mr. Simpleton retorts: "But my triangles had two sides of the one equal to two sides of the other, *therefore* the contained angles must be equal."

To such folly no answer is necessary. But it is a piece of folly that a large number of beginners imitate.

Caution. Care must be taken in drawing the third conclusion. Look carefully to see that the remaining angles which you assert to be equal are *corresponding* angles.

Corresponding angles are those which are opposite to the equal sides.

This mistake is most frequently made when the triangles are *reversed* in position, as in the accompanying figure.



[N.B. If a triangle is cut out in paper, its reverse is obtained by turning the paper over with its face downwards.]

In the triangles ABC, LMN, if it is given that AB=ML, BC=MN, and the angle ABC=the angle LMN, then the angle A corresponds to the angle L, because they are opposite to the equal sides BC, MN. The angle C corresponds to the angle N.

Questions.

1. In the triangles ABC, LMN it is given that AB = LM and BC = MN. Does it follow that the angle ABC is equal to the angle LMN?

2. But if the angle ABC is also given equal to the angle LMN, what conclusions do we get?
Which angle in the triangle LMN will be equal to the angle BCA?

3. Again, if in these triangles ABC, LMN it was given that the sides AB, CA are equal to the sides LM, NL respectively, and the angle BCA equal to the angle MNL, does it follow that the side BC equals the side MN?

4. If in Figure 1, EL, FM are joined, and it is given that EB = FD, BL = DM, and the angle EBL = the angle FDM, what conclusions do we draw?

5. In Figure 2, OB and OD are radii of the circle, and are therefore equal. Have the triangles OEB, OED two sides of the one equal to two sides of the other? If so, what angles must also be given equal, that we may be able to prove the triangles equal in all respects?

6. In Figure 4, if AD and BC are joined, the triangles AOD, BOC have the sides AO, OD equal to the sides BO, OC, each to each. Are the contained angles equal? If so, we can prove the triangles equal in all respects. What angles in the triangle BOC will be equal to the angles OAD, ODA respectively?

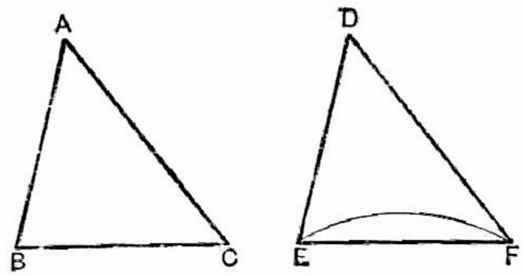
7. If in Figure 2, we join AC, the two sides AO, OC of the triangle AOC are equal to the two sides BO, OD of the triangle BOD. Are we able to draw the conclusion that AC is equal to BD?

8. If in Figure 3, GE and FA are joined, the triangles FGE, FGA are formed. Have these triangles two sides of the one equal to two sides of the other? If so, what angles do these equal pairs of sides contain? Are these contained angles equal? If so, what conclusions do we draw?

PROPOSITION 4. THEOREM.

General Enunciation.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal, then shall their bases or third sides be equal, and the triangles shall be equal in area, and their remaining angles shall be equal, each to each, namely, those to which the equal sides are opposite: that is to say, the triangles shall be equal in all respects.



Particular Enunciation.

Let **ABC**, **DEF** be two triangles, which have
the side **AB** = the side **DE**,
and the side **AC** = the side **DF**,
and the contained angle **BAC** = the contained angle **EDF**.

Then shall the base **BC** be equal to the base **EF**, and the triangle **ABC** shall be equal to the triangle **DEF** in area; and the remaining angles shall be equal, each to each, to which the equal sides are opposite, namely,
the angle **ABC** to the angle **DEF**,
and the angle **ACB** to the angle **DFE**.

Proof.

For if the triangle **ABC** be applied to the triangle **DEF**,
so that the point **A** may be on the point **D**,
and the straight line **AB** along the straight line **DE**,
then because **AB** is equal to **DE**, Hyp.
therefore the point **B** must coincide with the point **E**.

And because **AB** falls along **DE**,
and the angle **BAC** is equal to the angle **EDF**, Hyp.
therefore **AC** must fall along **DF**.

And because **AC** is equal to **DF**,
therefore the point **C** must coincide with the point **F**.

Then **B** coinciding with **E**, and **C** with **F**,
the base **BC** must coincide with the base **EF**;
for if not, two straight lines would enclose a space; which is impossible. Ax. 10.

Thus the base **BC** coincides with the base **EF**, and is therefore equal to it. Ax. 8.

And the triangle **ABC** coincides with the triangle **DEF**, and is therefore equal to it in area. Ax. 8.

And the remaining angles of the one coincide with the remaining angles of the other, and are therefore equal to them, namely,

the angle **ABC** to the angle **DEF**,
and the angle **ACB** to the angle **DFE**.

That is, the triangles are equal in all respects. Q.E.D

CHAPTER V.

In the accompanying figure OA, OB *are radii of a circle and* OP *is the bisector of the angle* AOB. *Prove that* AP *equals* PB.

[The bisector of an angle is that line which divides it into two equal angles.]

We see that the lines AP, PB, which we wish to prove equal, are sides of the triangles AOP, POB, and these triangles *appear* to be equal triangles reversed in position.

But it is necessary to *prove* them equal.

It will be found very helpful to draw figures accurately, for the appearance often helps to suggest a method of proof. But remember that appearance is not proof, and that a reason must be found for each step of the proof.

Firstly, have these triangles any equal sides?

Yes. OA equals OB, because they are radii of the same circle.

Can we state that AP is equal to BP?

Not at present, for this is the very thing we have to prove.

Look again. OP belongs to both triangles. And so we have the two sides AO, OP of the triangle AOP equal to the two sides BO, OP of the triangle BOP, each to each.

Now AO, OP contain the angle AOP,

and BO, OP contain the angle BOP.

Are these contained angles equal?

We must not, like Mr. Simpleton, say that they are equal, because they are contained angles.

But is there not some reason why they must be equal? or have they not been *given* equal?

Let us look at what was given. Have we made use of everything? What does "bisector" signify? Why, that the angle AOP equals the angle BOP.

We have therefore got two sides AO, OP and their contained angle AOP equal to two sides BO, OP and their contained angle BOP.

And so we conclude that the third side AP equals the third side BP.

This is written out as follows:

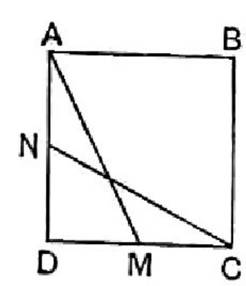
In the triangles AOP, BOP,

because $\begin{cases} \text{AO} = \text{BO, because they are radii of the same circle,} \\ \text{OP is common,} \\ \text{the contd. angle AOP} = \text{the contd. angle BOP (given);} \end{cases}$

therefore the third side AP = the third side BP. Q.E.D.

Another Example.

If N, M *are the middle points of the sides* AD, CD *of the square* ABCD. *Join* AM, NC *and prove them equal, and find angles equal to the angles* DAM, DMA (see Figure 1).



In the triangles ADM, CDN,

because $\begin{cases} \text{AD} = \text{DC, being sides of a square,} \\ \text{DM} = \text{DN, being halves of the sides,} \\ \text{and the contained angle at D is common;} \end{cases}$

therefore AM = CN,

and the angle DAM = the angle DCN,

and the angle DMA = the angle DNC. I. 4.

Q.E.D.

HINTS FOR PROVING TWO TRIANGLES EQUAL.

(1) Find two sides of the one which are equal to two sides of the other, each to each, and be careful to have a sufficient reason for stating that they are equal.

(2) Find the angles which these pairs of sides contain.

(3) Find a reason why these contained angles should be equal.

If these contained angles are not known or cannot be proved to be equal, your knowledge of Euclid is not yet in general sufficient to enable you to prove the triangles equal.

Exercise.

The student is advised to draw a separate figure in each case, marking only the lines and points that are required for the question in hand.

1. If ABCD is a square, and K, the middle point of AB, is joined to D and C, prove that KD = KC (see Figure 1).

2. If K, N, M are the middle points of the sides AB, AD, DC of a square ABCD, and KN, NM are joined, prove that KN = NM (see Figure 1).

3. If AOB is a diameter of a circle, and OC a radius at right angles to AB, prove that AC = CB (see Figure 2).

4. Two points B and D on the circumference of a circle are joined. OE, the bisector of the angle BOD, cuts BD in E. Prove that E is the middle point of BD (see Figure 2).

5. From M, the middle point of a straight line AB, another straight line is drawn at right angles to AB. If P is any point in this line, prove that PA = PB.

6. If L and M are the middle points of the sides BC, CD of the square ABCD, prove that BM = LD, and find an angle equal to the angle MBC (see Figure 1).

7. In Figure 1, if NC and LD are joined, prove that NC = LD, and find an angle equal to the angle LDC.

8. ABCD, AEFG are two squares with common angle at A. Join ED, BG and prove them equal (see Figure 3). What angle in this figure is equal to the angle GBA?

9. A circle, whose centre is A, cuts the sides AB, AD of the square ABCD in E and F. Join FB, DE and prove them equal (see Figure 1).

10. ABCDE is a five-sided figure, having all its sides equal and all its angles equal. Join AD, BD and prove them equal.
Also join CA, EB and prove them equal (see Figure 5).

11. In Figure 1, prove that FC = EC.

12. In Figure 3, prove CE = CG.

13. In Figure 1, prove EL = FM.

14. In Figure 4, prove QR = BC. [Note that it is given that the angle QOP = the angle AOB.]

15. In Figure 4, prove (1) AD = BC, (2) the angle ODA = the angle OCB, and (3) the angle OAD = the angle OBC.

16. Using what has been proved in question 15, prove in Figure 4 that (1) the angle ABC = the angle BAD, and (2) the angle BAC = the angle ABD.

17. Hence prove that the angle OAB = the angle OBA by Axiom 3.

If after practice in the previous questions the pupil has been able to do questions 15, 16 and 17 by himself, he has crossed the formidable "Pons Asinorum."

If he cannot do so entirely unaided he may possibly be able to overcome the difficulties with a few hints.

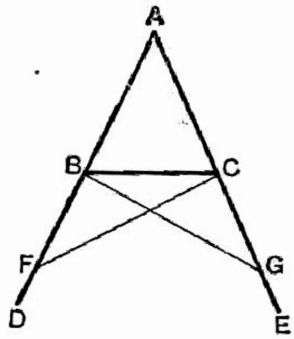
In question 17 you are asked to prove that *the angles at the base of the isosceles triangle* OAB *are equal to one another*, and in question 16 that *the angles at the other side of the base are equal to one another.*

The proofs of questions 15, 16, 17 should be written out in full. If the pupil has been able to do so, he will have little difficulty in learning Euclid's next proposition.

PROPOSITION 5. THEOREM.

General Enunciation.

The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced, the angles on the other side of the base shall also be equal to one another.



Particular Enunciation.

Let **ABC** be an isosceles triangle, having the side **AB** equal to the side **AC**, and let the straight lines **AB, AC** be produced to **D** and **E** :

then shall the angle **ABC** be equal to the angle **ACB**,
and the angle **CBD** to the angle **BCE.**

Construction.

In **BD** take any point **F** ;
and from **AE** the greater cut off **AG** equal to **AF** the less. I. 3.
Join **FC, GB.**

Proof.

Then in the triangles **FAC, GAB,**

because
- **FA=GA,** Constr.
- **AC=AB,** Hyp.
- also the contained angle at **A** is common to both triangles ;

therefore the triangle **FAC** is equal to the triangle **GAB** in all respects ; I. 4.

that is, the base **FC**=the base **GB,**
and the angle **ACF**=the angle **ABG,**
also the angle **AFC**=the angle **AGB.**

Again, because the whole **AF** = the whole **AG**, Constr.
and the part **AB** = the part **AC**, Hyp.
therefore the remainder **BF** = the remainder **CG**. Ax. 3.

Then in the two triangles **BFC, CGB**,

because { **BF** = **CG**, Proved.
FC = **GB**, Proved.
also the contained angle **BFC** = the contained angle **CGB**; Proved. }

therefore the triangles **BFC, CGB** are equal in all respects;
so that the angle **FBC** = the angle **GCB**,
and the angle **BCF** = the angle **CBG**, I. 4.

Now it has been shown that the whole angle **ABG** is equal to whole angle **ACF**,
and that the parts of these, namely, the angles **CBG, BCF** are also equal;
therefore the remaining angle **ABC** is equal to the remaining angle **ACB**;
and these are the angles at the base of the triangle **ABC**.

Also it has been shown that the angle **FBC** is equal to the angle **GCB**;
and these are the angles on the other side of the base. Q.E.D.

Exercise on Isosceles Triangles.

1. In Figure 1, if BD, EF, KN are joined, prove that
   (1) The angle ABD = the angle ADB.
   (2) The angle AKN = the angle ANK.
   (3) The angle AEF = the angle AFE.

2. In Figure 2, if AC, CB are joined, prove that
   (1) The angle OAC = the angle OCA.
   (2) The angle ACB = the sum of the angles CAB, CBA.
   (3) The angle OBD = the angle ODB.
   (4) The sum of the angles OCB, ODB equals the angle CBD.

3. In Figure 5, join GE and prove that the angle AGE equals the angle AEG.

4. Prove that all the angles of an equilateral triangle are equal.

5. ABC is an equilateral, and ABD an isosceles triangle on the same base AB. Prove that the angle CAD equals the angle CBD, whether the triangles are on the same side or on opposite sides of AB.

In the following questions, the equality of the triangles is proved with the help of the property of isosceles triangles.

6. If X, Y are the middle points of the equal sides AB, AC of the isosceles triangle ABC, prove that CX equals BY.

7. E, F are points on the base BC of the isosceles triangle ABC, such that BE equals CF. Prove that AE equals AF.

8. M is the middle point of the base BC of the isosceles triangle ABC. Prove that AM is at right angles to BC (see Definition 7).

9. L, M, N are the middle points of the base BC and the sides AC, AB of the isosceles triangle ABC. Prove that LM equals LN.

10. The equal sides BA, CA of an isosceles triangle BAC are produced beyond the vertex A to the points E and F, so that AE is equal to AF. Join FB and EC and prove them to be equal.

CHAPTER VI.

REDUCTIO AD ABSURDUM.

The following statement would be accepted as true:

If rain is falling, there are clouds overhead.

It does not follow that,

If there are clouds overhead, rain is falling.

The second statement is called the **converse** of the first.

One proposition is said to be the **converse** of another, when the hypothesis of each is the conclusion of the other.

In the above illustration, 'rain is falling' is the hypothesis of the first proposition and the conclusion of the second; 'there are clouds overhead' is the conclusion of the first and the hypothesis of the second proposition.

Therefore the second proposition is the converse of the first. But the first proposition is true, while the second is not true.

We see that the converse proposition of a proposition already proved is not necessarily true. However, it often happens that a converse proposition is true, but it must be proved before it can be accepted as true.

The converse of the proposition,

If two sides of a triangle be equal to one another, then the angles which are subtended by the equal sides shall be also equal to one another, is,

If two angles of a triangle be equal to one another, then the sides which subtend the equal angles shall be also equal to one another.

We have now to find a method to prove this converse proposition true.

There are two methods by which converse propositions may frequently be proved, either by (1) **the exhaustive method,** or (2) a method called "**reductio ad absurdum,**" which literally translated means "a reducing to absurdity."

The exhaustive method is not used in the first twelve propositions, so we will not discuss it. Let us try to understand the second method.

"**Reductio ad absurdum**" is used when we wish to prove the truth of a proposition by showing that the opposite of that proposition cannot possibly be true.

For example :

> A thing is *either* (1) a ship *or* (2) not a ship.
> An article is *either* (1) white *or* (2) not white.
> An animal *either* (1) has a tail *or* (2) has no tail.
> Two things are *either* (1) equal *or* (2) unequal.

Now in each case *either* (1) *or* (2) must be true.

If we *assume* that (2) is true, and by reasoning from that assumption arrive at an absurdity, we conclude *either* that our reasoning has been incorrect *or* our assumption incorrect.

But if no flaw can be found in our reasoning, our assumption viz. that (2) is true, must have been incorrect.

Therefore supposing that (1) or (2) must be true, and that we have proved that (2) cannot be true, it follows that (1) must be true.

Hence in a "reductio ad absurdum" *we assume as true the opposite of the fact that we wish to prove.*

If with that assumption we arrive at an absurdity, we conclude that the fact, that we wish to prove true, is true.

Let us take this simple case, which Euclid proves in Book III. :

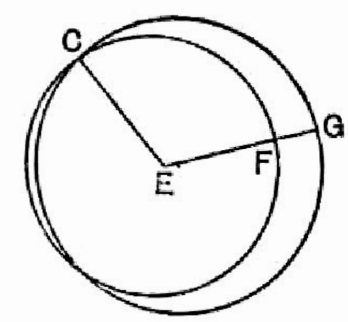
Two circles which cut one another cannot have a common centre.

We assume as true the opposite of the fact that we wish to prove.

Therefore we assume that they *have* a common centre.

If possible let the two circles have a common centre E.

Join E to C, the point where the circles cut, and from E draw a straight line to meet the circumferences in F and G.



We can prove that EF and EG are each equal to EC, and therefore that EF is equal to EG.

But this is absurd;
and so, since the circles either (1) have a common centre or (2) do not have a common centre, and since (1) has been proved untrue, we see that (2) must be true.

There is considerable difficulty in dealing with this method, arising from the fact that we are reasoning from a false assumption and that the reasoning is not borne out by the appearance of the figure.

For instance, in the proof just given, it goes against the grain to make the statement that EF equals EC, when our eye contradicts the fact.

And so it requires great care to reason correctly on a false assumption (or hypothesis as it is called) contrary to the evidence of our eyes.

CONVERSE OF THE "PONS ASINORUM."

We have seen that if two sides of a triangle are equal, the angles opposite to them are equal.

We have to prove that *if two angles of a triangle are equal, the sides opposite to them are equal.*

We employ "reductio ad absurdum."

We assume as true the opposite of the fact that we wish to prove. Therefore we assume that the sides are unequal.

If so, one side must be greater than the other.

Let ABC be a triangle having the angle ABC equal to the angle ACB (see Figure on the opposite page).

We have to prove that AC, which is opposite to the angle ABC, is equal to AB, which is opposite to the angle ACB.

If AC is not equal to AB, let AB be the greater. Then from BA the greater, a part BD can be cut off equal to AC the less.

Here arises a difficulty, for to the eye BD is obviously not equal to AC, and in the subsequent reasoning we are apt to forget that we have assumed it to be so.

We join DC.

Now if we can prove that the triangle DBC is equal in area to the triangle ABC, we shall have arrived at an absurdity.

In investigating these triangles we must be especially careful to recall what has been *given*, and what has been *assumed*.

We have

$$\begin{cases} \text{BD} = \text{AC by assumption,} \\ \text{BC common,} \\ \text{the contd. angle DBC} = \text{the contd. angle ACB (given),} \end{cases}$$

and so we see that the triangles are equal in area and this is absurd.

Now, as our reasoning has been correct, our assumption must have been wrong. But the lines AB, AC must be either equal or unequal, and we have shown that they cannot be unequal. Therefore they must be equal.

Exercise on Proposition 6.

1. If ABCD is a square, and the bisectors AO, BO of the angles at A and B meet in O, prove that AO = BO.

2. Prove that triangles which have all their angles equal have all their sides equal, or in other words that equiangular triangles are also equilateral.

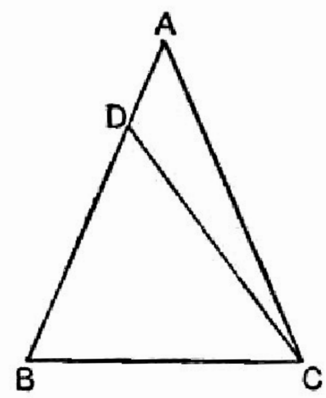
3. AO, BO, the bisectors of two angles of an equilateral triangle, meet in O. Prove that AO = BO.

4. At the extremities of the base AB of an isosceles triangle ABC, AP is drawn perpendicular to AC and BP is drawn perpendicular to CB. These perpendiculars meet in P. Prove that PA = PB.

PROPOSITION 6. THEOREM.

General Enunciation.

If two angles of a triangle be equal to one another, then the sides also which subtend, or are opposite to, the equal angles, shall be equal to one another.



Particular Enunciation.

Let **ABC** be a triangle, having the angle **ABC** equal to the angle **ACB** :

then shall the side **AC** be equal to the side **AB**.

Construction.

For if **AC** be not equal to **AB**,
one of them must be greater than the other.
If possible, let **AB** be the greater ;
and from it cut off **BD** equal to **AC**. I. 3.
Join **DC**.

Proof.

Then in the triangles **DBC, ACB,**

because
- **DB**=**AC**, Constr.
- **BC** is common to both,
- and the contained angle **DBC**=the contained angle **ACB** ; Hyp.

therefore the triangle **DBC** is equal in area to the triangle **ACB**, I. 4.

the part equal to the whole ; which is absurd. Ax. 9.
Therefore **AB** is not unequal to **AC** ;
that is, **AB** is equal to **AC**. Q.E.D.

CHAPTER VII.

In Figure 1, on page 83, if the lines FB, FE, EN are joined, let us prove that the angle AFB is greater than the angle AEN.

We see that the angle AFE is equal to the angle AEF, because AE is equal to AF.

Now the angle AFE is only a part of the angle AFB, while the angle AEF contains the angle AEN.

Therefore our proof runs as follows :

> The angle AFB is greater than its part the angle AFE.
> But the angle AFE = the angle AEF.
> Therefore the angle AFB is also greater than the angle AEF.
> Much greater therefore is the angle AFB than the angle AEN.

Exercise.

1. In Figure 3, if K is a point between A and G, prove that the angle AGB is greater than the angle AEK.

2. In Figure 5, prove that the angle CBA is greater than the angle FAB.

INTRODUCTION TO PROPOSITION 7.

In the accompanying figure, A is the centre of the semi-circle, C, D any points on its circumference, and B any other point on the diameter.

The triangles ACB, ADB have their sides terminated at A equal, *i.e.* the sides AC and AD.

But however we draw the figure, so long as C and D are on the same side of AB, we cannot get two triangles ACB, ADB on the same base AB such that AC, AD are equal, and *also* BC, BD equal.

We can prove this by "reductio ad absurdum."

We assume as true that BC, BD can be *also* equal.

If so, the angle BDC *is equal* to the angle BCD, for they will be the angles at the base of the isosceles triangle BCD.

But we can prove that the angle BDC *is greater than* the angle BCD.

In the first place, let the vertices C and D be each without the other triangle, as in Figure 1.

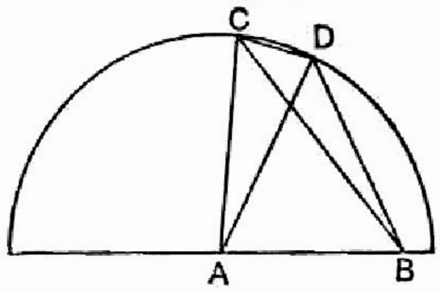


Fig. 1.

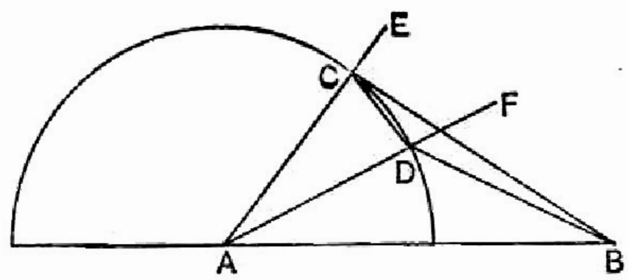


Fig. 2.

For the angle BDC is greater than its part the angle ADC, and is therefore also greater than the angle ACD, and therefore much greater than the angle BCD.

But one angle cannot at the same time be greater than and equal to another angle.

Hence we have an absurdity.

And so we see that BC and BD cannot be *also* equal.

In the second place, let D be within the triangle ACB as in Figure 2.

Let AC, AD be produced to E and F.

Then the angles ECD, FDC on the far side of the base of the isosceles triangle ACD are equal.

Since we have assumed that BC is equal to BD, therefore the angle BDC *is equal to* the angle BCD.

But we can prove that the angle BDC *is greater than* the angle BCD.

For the angle BDC is greater than its part the angle FDC, and therefore greater than the angle ECD, and therefore much greater than BCD.

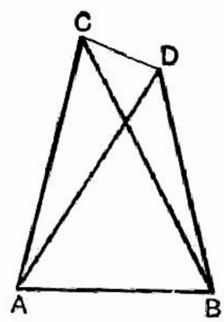
And so, in either case, we get an absurdity.

Therefore in the triangles ACB, ADB if the sides AC, AD terminated at A are equal, then BC, BD terminated at B cannot *also* be equal.

PROPOSITION 7. THEOREM.

General Enunciation.

On the same base, and on the same side of it, there cannot be two triangles having their sides which are terminated at one extremity of the base equal to one another, and likewise those which are terminated at the other extremity equal to one another.



Particular Enunciation.

If it be possible, on the same base **AB**, and on the same side of it, let there be two triangles **ACB**, **ADB**, having their sides **AC**, **AD**, which are terminated at **A**, equal to one another, and likewise their sides **BC**, **BD**, which are terminated at **B**, equal to one another.

CASE I. When the vertex of each triangle is without the other triangle.

Construction. Join **CD**.

Proof. Then in the triangle **ACD**,

because **AC**=**AD**, Hyp.

therefore the angle **ACD**=the angle **ADC**. I. 5.

But the whole angle **ACD** is greater than its part, the angle **BCD**,

therefore also the angle **ADC** is greater than the angle **BCD**;

still more then is the angle **BDC** greater than the angle **BCD**.

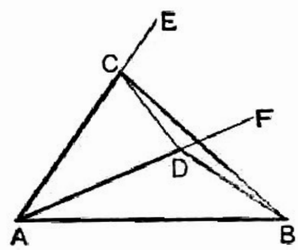
Again, in the triangle **BCD**,

because **BC**=**BD**, Hyp.

therefore the angle **BDC**=the angle **BCD**: I. 5.

but it was shown to be greater; which is impossible.

CASE II. When one of the vertices, as **D**, is within the other triangle **ACB**.



Construction. As before, join **CD** ;
and produce **AC**, **AD** to **E** and **F**.

Proof. Then in the triangle **ACD**,
because **AC**=**AD**, Hyp.
therefore the angles **ECD**, **FDC**, on the other side of the base, are equal to one another. I. 5.

But the angle **ECD** is greater than its part, the angle **BCD** ;
therefore the angle **FDC** is also greater than the angle **BCD** :
still more then is the angle **BDC** greater than the angle **BCD**.

Again, in the triangle **BCD**,
because **BC**=**BD**, Hyp.
therefore the angle **BDC**=the angle **BCD** : I. 5.
but it has been shown to be greater ; which is impossible.

The case in which the vertex of one triangle is on a side of the other needs no demonstration.

Therefore **AC** cannot be equal to **AD**, and *at the same time* **BC** equal to **BD**. Q.E.D.

NOTE. The sides AC, AD are called *conterminous* sides : similarly the sides BC, BD are conterminous sides.

ON THE EQUALITY OF TRIANGLES.

We have already shown in Proposition 4 that if certain conditions are given, we can prove that two triangles are identically equal.

The conditions given in Proposition 4 are not the *only* conditions which will enable us to prove two triangles identically equal.

We are now going to show that if two triangles have *all* the sides of one equal to *all* the sides of the other, these triangles are identically equal.

And this proof is effected by means of the proposition last proved.

Let the triangles ABC, DEF on the opposite page be such that

$$AB = DE, \quad BC = EF, \quad CA = FD.$$

We can take up the triangle ABC and place BC upon EF, for BC is equal to EF.

Then if we lay the triangle ABC down, it will fall exactly on the triangle DEF.

For suppose that it falls in some other position, such as GEF.

Then we have the two triangles DEF, GEF on the same base and on the same side of it with their *conterminous* sides equal, viz., DE = GE, and *also* DF = GF.

But this is impossible, as we have just shown.

Therefore we see that the triangle ABC falls exactly on the triangle DEF, and therefore the whole triangles are equal in all respects, and the corresponding angles of the two triangles are equal.

Important Note.

We have here proved that the *triangles* are equal in all respects. Euclid in Proposition 8 confines himself to proving the *angles* equal.

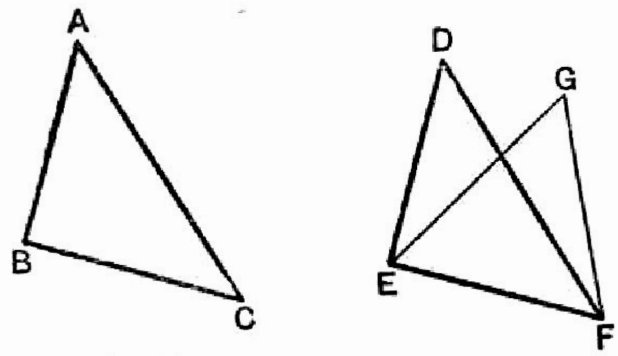
A *corollary* has therefore to be added to his proposition.

A corollary is a statement, the truth of which follows readily from an established proposition.

PROPOSITION 8. THEOREM.

General Enunciation.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, then the angle which is contained by the two sides of the one shall be equal to the angle which is contained by the two sides of the other.



Particular Enunciation.

Let **ABC, DEF** be two triangles having the two sides **BA, AC** equal to the two sides **ED, DF**, each to each, namely, **BA** to **ED**, and **AC** to **DF**, and also the base **BC** equal to the base **EF**:

then shall the angle **BAC** be equal to the angle **EDF.**

Proof.

For if the triangle **ABC** be applied to the triangle **DEF**, so that the point **B** may be on **E**, and the straight line **BC** along **EF**;

then because **BC=EF**, Hyp.

therefore the point **C** must coincide with the point **F**.

Then **BC** coinciding with **EF**,

it follows that **BA** and **AC** must coincide with **ED** and **DF**: for, if not, they would have a different situation, as **EG, GF**: then, on the same base and on the same side of it there would be two triangles having their *conterminous* sides equal.

But this is impossible. I. 7.

Therefore the sides **BA, AC** coincide with the sides **ED, DF**. That is, the angle **BAC** coincides with the angle **EDF**, and is therefore equal to it. Ax. 8.

Q.E.D.

Corollary. *If in two triangles the three sides of the one are equal to the three sides of the other, each to each, then the triangles are equal in all respects.*

CHAPTER VIII.

We have now two ways of proving two triangles to be equal in all respects.

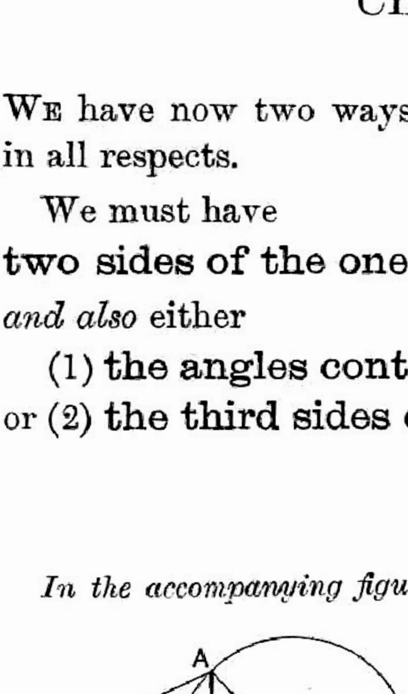
We must have

two sides of the one equal to two sides of the other,
and also either

(1) **the angles contained by these equal sides equal,**
or (2) **the third sides equal.**

Example.

In the accompanying figure A *and* B *are any two points on the circumference of the circle, and* ABF *is an isosceles triangle described on the base* AB. *Prove that the angle* ACF *is equal to the angle* BCF.



It happens that this can be proved in either of the ways mentioned above. We give both proofs.

I. Because BC = AC, therefore the angle CAB = the angle CBA,
and because FB = FA, therefore the angle FAB = the angle FBA.

Therefore the whole angle FAC = the whole angle FBC.

Then in the triangles FAC, FBC,

because $\begin{cases} \text{FA} = \text{FB}, \\ \text{AC} = \text{CB}, \\ \text{and the contained angle FAC} = \text{the contained angle FBC}; \end{cases}$

therefore the triangles are equal in all respects, I. 4.

and so the angle ACF = the angle BCF.

Or, II. In the triangles FAC, FBC,

because $\begin{cases} FA = FB, \\ AC = CB, \\ FC \text{ is common to both triangles;} \end{cases}$

therefore the angle ACF = the angle BCF. I. 8.

It will not often happen that each of these two ways can be employed. In general we must look out for one or the other.

Further Exercise on Equal Triangles.

1. A, B are two points on the circumference of a circle whose centre is C, and M is the middle point of AB. If CM are joined, prove that (1) the angle ACM equals the angle BCM, (2) the angle AMC equals the angle CMB, (3) that CM is perpendicular to AB.

2. ABCD is a four-sided figure having AB equal to AD and BC equal to CD. Prove that the angle ABC is equal to the angle ADC, and that AC bisects both the angles BAD and BCD.

3. Two circles whose centres are O and C cut each other in A and B, prove that the angle OAC equals the angle OBC.

4. If A, B, C, D are four points on the circumference of a circle whose centre is O and such that the line AB is equal to the line CD, prove that the angle AOB is equal to the angle COD.

5. If the four-sided figure ABCD has its opposite sides AB, CD equal, and also BC, DA equal, prove that the opposite angles are also equal.

6. ABC is an isosceles triangle having AB equal to AC, and the angles at B and C are bisected by straight lines which meet at O. Show that OA bisects the angle BAC.

7. MLN is any angle and S and T are points in the arms LM, LN such that LS equals LT. On ST on the side remote from L an equilateral triangle SPT is described. Prove that LP bisects the angle MLN.

8. On the line PQ an equilateral triangle PRQ is drawn, and RM the bisector of the angle PRQ cuts PQ in M. Prove that M is the middle point of PQ.

9. P is any point in the line AB; M and N are points in AB such that PM is equal to PN; Q is the vertex of an equilateral triangle described on MN. Prove that PQ is perpendicular to AB.

10. AB is any line and P any point without it; a circle with centre P is drawn to cut AB in Q and R; P is joined to M the middle point of QR. Prove that PM is perpendicular to AB.

ON CERTAIN IMPORTANT CONSTRUCTIONS.

In the above exercise, in question 7 the construction has been given whereby *the angle* MLN *may be bisected;*

In question 8 the construction whereby *the line* PQ *may be bisected;*

In question 9 the construction whereby *a perpendicular may be drawn from* P *to* AB, *where* P *lies on* AB;

In question 10 the construction whereby *a perpendicular may be drawn from* P *to* AB, *where* P *lies without* AB.

In each case the proof has been asked for.

These constructions are important and should be carefully studied.

They are Euclid's constructions for

(1) Bisecting an angle;

(2) Bisecting a straight line;

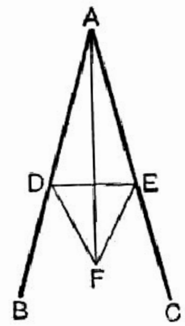
(3) Drawing a perpendicular to a given straight line from a given point in it;

(4) Drawing a perpendicular to a given straight line from a given point without it.

PROPOSITION 9. PROBLEM.

General Enunciation.

To bisect a given angle, that is, to divide it into two equal parts.



Particular Enunciation.

Let **BAC** be the given angle ;
it is required to bisect it.

Construction. In **AB** take any point **D** ;
and from **AC** cut off **AE** equal to **AD**. I. 3.
Join **DE**.
and on **DE**, on the side remote from **A**, describe an equilateral triangle **DEF**. I. 1.
Join **AF**.

Then shall the straight line **AF** bisect the angle **BAC**.

Proof. For in the two triangles **DAF**, **EAF**,

because { **DA** = **EA**, Constr.
and **AF** is common to both ;
and **DF** = **EF** ; Def. 19.

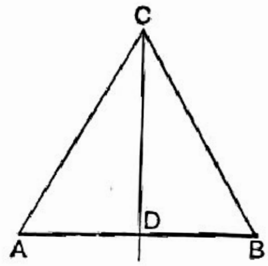
therefore the angle **DAF** = the angle **EAF**. I. 8.

Therefore the given angle **BAC** is bisected by the straight line **AF**. Q.E.F.

PROPOSITION 10. PROBLEM.

General Enunciation.

To bisect a given finite straight line, that is, to divide it into two equal parts.



Particular Enunciation.

Let **AB** be the given straight line:
it is required to divide it into two equal parts.

Construction.

On **AB** describe an equilateral triangle **ABC**, I. 1.
and bisect the angle **ACB** by the straight line **CD**, I. 9.
and let **CD** cut **AB** in **D**.

Proof.

For in the triangles **ACD**, **BCD**,

because { **AC**=**BC**, Def. 19.
and **CD** is common to both;
also the contained angle **ACD**=the contained angle **BCD**; Constr. }

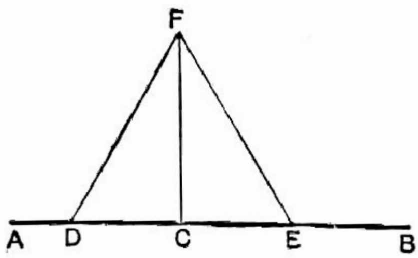
therefore the triangles are equal in all respects:
so that the base **AD**=the base **BD**. I. 4.
Therefore the straight line **AB** is bisected at the point **D**.

Q.E.F.

PROPOSITION 11. PROBLEM.

General Enunciation.

To draw a straight line at right angles to a given finite straight line from a given point in the same.



Particular Enunciation.

Let **AB** be the given straight line and **C** the given point in it. It is required to draw from the point **C** a straight line at right angles to **AB**.

Construction. In **AC** take any point **D**,
and from **CB** cut off **CE** equal to **CD**. I. 3.
On **DE** describe the equilateral triangle **DFE**. I. 1.
Join **CF**.
Then shall the straight line **CF** be at right angles to **AB**.

Proof. For in the triangles **DCF, ECF,**

because $\left\{\begin{array}{l} \mathbf{DC}=\mathbf{EC}, \quad \text{Constr.} \\ \text{and } \mathbf{CF} \text{ is common to both,} \\ \text{and } \mathbf{DF}=\mathbf{EF}; \quad \text{Def. 19.} \end{array}\right.$

therefore the angle **DCF**=the angle **ECF**, I. 8.
and these are adjacent angles.

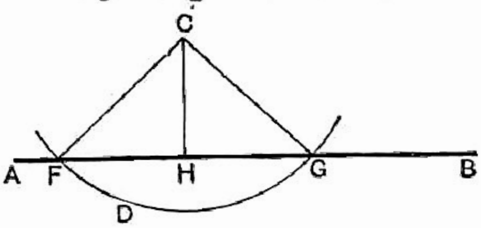
But when a straight line, standing on another straight line, makes the adjacent angles equal to one another, each of these angles is called a right angle ; Def. 7.

therefore each of the angles **DCF, ECF** is a right angle.
Therefore **CF** is at right angles to **AB**,
and has been drawn from a point **C** in it. Q.E.F.

PROPOSITION 12. PROBLEM.

General Enunciation.

To draw a straight line perpendicular to a given straight line of unlimited length from a given point without it.



Particular Enunciation.

Let **AB** be the given straight line which may be produced in either direction, and let **C** be the given point without it.

It is required to draw from the point **C** a straight line perpendicular to **AB**.

Construction.

On the side of **AB** remote from **C** take any point **D**; and from centre **C**, with radius **CD**, describe the circle **FDG**, meeting **AB** at **F** and **G**.

Bisect **FG** at **H**; I. 10.

and join **CH**.

Then shall the straight line **CH** be perpendicular to **AB**.

Join **CF** and **CG**.

Proof. Then in the triangles **FHC**, **GHC**,

because { **FH** = **GH**, Constr.
and **HC** is common to both;
and **CF** = **CG**, being radii of the circle **FDG**;

therefore the angle **CHF** = the angle **CHG**; I. 8.

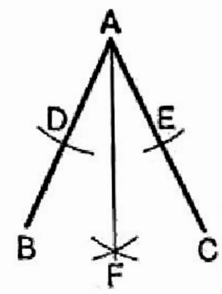
and these are adjacent angles.

But when a straight line, standing on another straight line, makes the adjacent angles equal to one another, each of these angles is called a right angle, and the straight line which stands on the other is called a perpendicular to it.

Therefore **CH** is a perpendicular drawn to the given straight line **AB** from the given point **C** without it. Q.E.F.

In *Geometrical Drawing* the following are the usual constructions given.

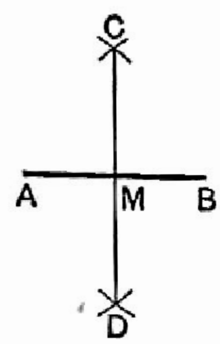
(1) *To bisect the angle* BAC.



With centre A and *any* radius, describe a circle cutting the arms in D and E. With radius *unchanged*, and centres D and E, describe two equal circles.

The line joining A to the point F where the circles intersect is the bisector.

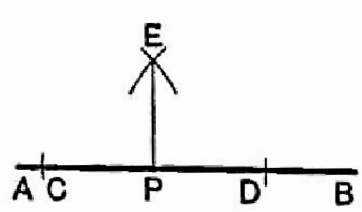
(2) *To bisect a line* AB.



With centres A and B, and *any* radius greater than half AB, describe equal circles intersecting in C and D.

The point M where CD cuts AB is the middle point of AB.

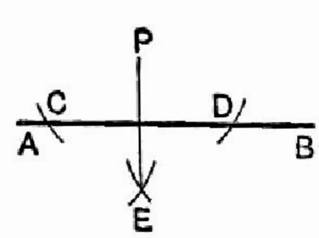
(3) *To erect a perpendicular from a given point* P *in the line* AB.



With centre P and *any* radius describe a circle, cutting AB in C and D. With centres C and D and *any* radius greater than PC, describe circles intersecting in E.

Then PE is the perpendicular.

(4) *To drop a perpendicular from the point* P *on the line* AB.



With centre P and *any* radius of sufficient length draw a circle cutting AB in C and D. With centres C and D and radius *unchanged*, draw equal circles intersecting in E.

Then PE will be perpendicular to AB.

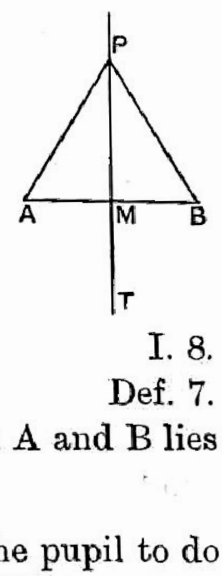
The difference between these and the Euclidean constructions lies in the fact that wherever it is possible circles of *any* radius are drawn instead of circles of some precise radius, such as are involved in drawing equilateral triangles. The reason for this is that time is wasted in adapting compasses to any precise length.

It is a good exercise to prove by Euclidean methods that these constructions are correct.

CHAPTER IX.

LOCI AND MISCELLANEOUS EXAMPLES.

To find the line on which all points lie which are equidistant from two given points.



Let AB be the two points, and let P be *any* point equidistant from A and B.

Join P to M the middle point of AB.

Then in the triangles AMP, BMP,

because $\begin{cases} AP = PB, & \text{Given.} \\ AM = MB, & \text{Constr.} \\ PM \text{ is common;} & \end{cases}$

therefore the angle PMA = the angle PMB. I. 8.

Therefore PM is perpendicular to AB. Def. 7.

Therefore *any* point P which is equidistant from A and B lies on the line which bisects AB at right angles.

It can be easily proved, and it may be left to the pupil to do so, that any point T on the line PM must be equidistant from A and B.

Wherefore we see that all points equidistant from A and B lie on the line PM, and that all points on this line are equidistant from A and B.

The line PM is called the *locus* of points equidistant from A and B.

Again, the *locus* of all points at a certain given distance from a fixed point is a circle having that point for its centre, and the given distance for radius.

Again, the *locus* of all points in a certain direction from a given point is a straight line through that point.

A **locus** is the path traced out by a point, which moves in such a way, that it will satisfy a certain condition or law.

Suppose that you are directing a friend to find a ball, and that you say that it is ten yards from a certain tree.

He knows then the locus of the ball, and if he walks round the tree at a distance of ten yards from it, he will find it. But the ball may have been in any one of an infinite number of positions.

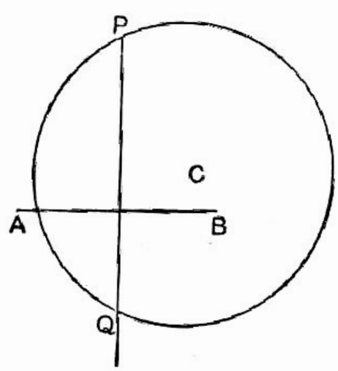
But if you are able to tell him that the ball also lies along a certain line, he can follow that line up, and it will cut the circle round the tree generally in two places, and he will be able to find the ball without search. You have given him two loci. He knows that the ball lies on a certain circle ; he also knows that it lies along a certain straight line. He concludes that it must lie at the *intersection* of these two loci. This is an instance of the principle of the **intersection of loci**.

There are any number of points that will *satisfy a certain condition*, or in other words that will lie on a certain locus.

But in general there are only one or two points that will satisfy two conditions, or in other words that will lie on each of two loci, viz. those points where the two loci intersect.

EXAMPLE.

Find a point equidistant from A *and* B, *and half-an-inch from the point* C.



The point required must lie on the perpendicular to AB through M, the middle point of AB, because it is equidistant from A and B.

Also it must lie on a circle whose centre is C and radius half-an-inch.

Therefore it must be one of the two points where these loci intersect, *i.e.* either at P or Q.

Therefore P and Q are points equidistant from A and B, and each half-an-inch from C.

Exercise.

1. Find a point in a straight line AB of unlimited length equidistant from two points P and Q.

2. In Figure 1 find a point equidistant from A and B, and distant CB from C.

3. Find a point equidistant from three points A, B, C, which are not in the same straight line.

4. In the side AB or in AB produced of a triangle ABC, find a point equidistant from B and C.

5. A town B is 6 miles due east of a town C. Find a point due north of C and 8 miles from B. (Scale $\frac{1}{4}$ in. to the mile.)

6. In Figure 3 describe a triangle on AD, having one side equal to AE and the other to DF.

7. Two houses A and B are on each side of a straight road. Find a point in it equidistant from A and B.

8. In Figure 5 describe an isosceles triangle on AG as base, having its vertex in AE.

9. A, B, C are 3 villages on a straight road, B is 2 miles north of A and 4 miles south of C. Find a point equidistant from A and C and 4 miles from B.

EQUALITY OF TRIANGLES.

We shall often find that when we want to prove two triangles equal in all respects the proof involves several preliminary steps.

Example.

To prove the triangle ABC *equal in all respects to the triangle* ABD *in Figure* 4.

All we know *without preliminary proof* is that AB is common. A preliminary proof is required to show that

(1) AC = BD, (2) BC = AD.

The whole proof will be written out as follows:

Because OC = OD, radii of the same circle,
and OA = OB, radii of the same circle,
therefore the remainder AC = the remainder BD. Ax. 3.

In the triangles OBC, OAD,

because $\begin{cases} \text{OB} = \text{OA}, \\ \text{OC} = \text{OD}, \\ \text{the contained angle AOB is common;} \end{cases}$

therefore the base BC = the base AD. I. 4.

In the triangles ABC, ABD,

because $\begin{cases} \text{AC} = \text{BD}, & \text{Proved.} \\ \text{BC} = \text{AD}. & \text{Proved.} \\ \text{AB is common;} \end{cases}$

therefore the triangles ABC, ABD are equal in all respects.

Q.E.D.

Exercise.

1. ABCD is a square. With centre A and radius AB a circle is drawn. AE is a radius drawn at right angles to AB and remote from AD. Prove that (1) CB = AE, (2) CA = BE.

2. To the corner A of a square hut ABCD a donkey is tethered by a rope, whose length is twice the length of a side of the hut. Draw a figure to show the ground over which the donkey can graze.

3. ABC is an isosceles and ABD an equilateral triangle on the same base AB. Prove that DC bisects the angle ACB whether the two triangles are on the same or on opposite sides of AB.

4. Draw a triangle having its sides equal in length to three given straight lines. Is this always possible?

5. BCDE is a square, and on BC as base an isosceles triangle ABC is described. Prove that the angle ABE equals the angle ACD and that AE equals AD.

6. On the arms BC, CE of an angle BCE, squares CBAD, CEFG are described on the sides remote from the angle. Prove that the angle BCG is equal to the angle DCE, and that BG is equal to DE.

7. Divide a given angle into four equal parts.

8. On AB one of the equal sides of an isosceles triangle ABC, an equilateral triangle ABD is drawn. Prove that the angle ADC is equal to the angle ACD.

9. Draw a rhombus given the position of two opposite angular points, and the length of the equal sides.

10. M and N are points in the diagonal AC of the square ABCD, so that AM = NC. Prove that the angle BMN is equal to the angle BNM.

11. In Figure 5 prove (1) EG = FC, (2) DG = DF.

12. In Figure 1 prove the angle ELK equal to the angle NMF.

13. In Figure 3 prove the angle GCF equal to the angle ECF.

14. Divide a straight line into four equal parts.

15. In Propositions 9, 10, 11 would the proof be affected, if isosceles triangles were drawn instead of equilateral triangles?

16. In Figure 4 prove the angle RQP equal to the angle DAB.

17. If AO, BO the bisectors of the angles A and B of the regular pentagon ABCDE meet in O, prove that AO = BO.

18. From A and B in Figure 5, AP is drawn perpendicular to AE, and BP perpendicular to BC. These perpendiculars meet in P. Prove that AP = PB.

19. Could the following construction be substituted for that in Prop. 10? On AB describe two equilateral triangles ABC, ABD. Join CD, cutting AB in M. Then M is the middle point of AB.

20. In Figure 4, if E is any point in OA, prove that the angle OBE is less than the angle OAD.

21. If in Figure 2, BA is produced to any point P, show that the triangles CAP, CBP have two sides and an angle of the one equal to two sides and an angle of the other. Why are not the triangles equal?

22. Draw a right-angled isosceles triangle having each of its equal sides one inch in length.

23. From a given point C in a straight line AB, make an angle equal to half a right angle.

THE DEFINITIONS, POSTULATES, AXIOMS, AND FIRST TWELVE PROPOSITIONS

OF

EUCLID'S ELEMENTS.

DEFINITIONS.

1. A **point** is that which has position, but no magnitude.

2. A **line** is that which has length without breadth.

3. A **straight line** is that which lies evenly between its extreme points.

4. A **surface** is that which has length and breadth, but no thickness.

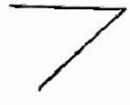
5. A **plane surface** is one in which any two points being taken, the straight line between them lies wholly in that surface.

6. A **plane angle** is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.

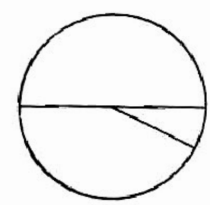
7. When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a **right angle**; and the straight line which stands on the other is called a **perpendicular** to it.

8. An **obtuse angle** is an angle which is greater than one right angle, but less than two right angles.

9. An **acute angle** is an angle which is less than a right angle.



10. Any portion of a plane surface bounded by one or more lines, straight or curved, is called a **plane figure**.



11. A **circle** is a plane figure contained by one line which is called the **circumference**, and as such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another: this point is called the **centre** of the circle.

A **radius** of a circle is a straight line drawn from the centre to the circumference.

12. A **diameter** of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

13. A **semicircle** is the figure bounded by a diameter of a circle and the part of the circumference cut off by the diameter.

14. A **segment** of a circle is the figure bounded by a straight line and the part of the circumference which it cuts off.

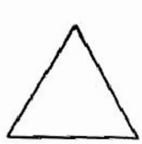
15. **Rectilineal figures** are those which are bounded by straight lines.

16. A **triangle** is a plane figure bounded by *three* straight lines.

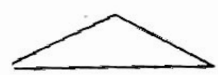
17. A **quadrilateral** is a plane figure bounded by *four* straight lines.

The straight line which joins opposite angular points in a quadrilateral is called a **diagonal**.

18. A **polygon** is a plane figure bounded by more than four straight lines.



19. An **equilateral triangle** is a triangle whose three sides are equal.



20. An **isosceles triangle** is a triangle two of whose sides are equal.

21. A **scalene triangle** is a triangle which has three unequal sides.

22. A **right-angled triangle** is a triangle which has a right angle.

23. An **obtuse-angled triangle** is a triangle which has an obtuse angle.

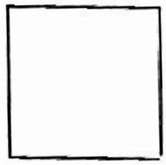
24. An **acute-angled triangle** is a triangle which has *three* acute angles.

25. **Parallel straight lines** are such as, being in the same plane, do not meet, however far they are produced in either direction.

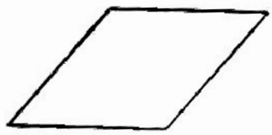
26. A **parallelogram** is a four-sided figure which has its opposite sides parallel.

27. A **rectangle** is a parallelogram which has one of its angles a right angle.

28. A **square** is a four-sided figure which has all its sides equal and all its angles right angles.



29. A **rhombus** is a four-sided figure which has all its sides equal, but its angles are not right angles.



30. A **trapezium** is a four-sided figure which has *two* of its sides parallel.

POSTULATES.

Let it be granted :

1. That a straight line may be drawn from any one point to any other point.

2. That a *finite*, *i.e.* a terminated straight line, may be produced to any length in that straight line.

3. That a circle may be described from any centre, at any distance from that centre, *i.e.* with a radius equal to a finite straight line drawn from the centre.

AXIOMS.

1. Things which are equal to the same thing are equal to one another.

2. If equals be added to equals, the wholes are equal.

3. If equals be taken from equals, the remainders are equal.

4. If equals be added to unequals, the wholes are unequal, the greater sum being that which includes the greater unequal.

5. If equals be taken from unequals, the remainders are unequal, the greater remainder being that which is left from the greater of the unequals.

6. Things which are double of the same thing, or of equal things, are equal to one another.

7. Things which are halves of the same thing, or of equal things, are equal to one another.

8. Magnitudes which can be made to coincide with one another are equal.

9. The whole is greater than its part.

10. Two straight lines cannot enclose a space.

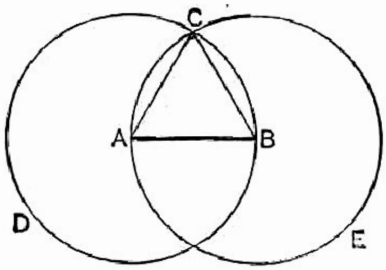
11. All right angles are equal.

12. If a straight line meet two straight lines so as to make the interior angles on one side of it together less than two right angles, these straight lines will meet if continually produced on the side on which are the angles which are together less than two right angles.

PROPOSITION 1. PROBLEM.

General Enunciation.

To describe an equilateral triangle on a given finite straight line.



Particular Enunciation.

Let **AB** be the given finite straight line.

It is required to describe an equilateral triangle on **AB.**

Construction.

From centre **A** with radius **AB**, describe the circle **BCD**.

From centre **B** with radius **BA**, describe the circle **ACE**.

Let the circles intersect in **C**.

Join **CA** and **CB**.

Then shall **ABC** be an equilateral triangle.

Proof. Because **A** is the centre of the circle **BCD,**

therefore **AC** = **AB,**

and because **B** is the centre of the circle **ACE,**

therefore **BC** = **BA**.

Therefore **AC** and **BC** are each equal to **AB.**

But things which are equal to the same thing are equal to one another. Ax. 1.

Therefore **AC** = **BC**.

Therefore **CA, AB, BC** are equal to one another.

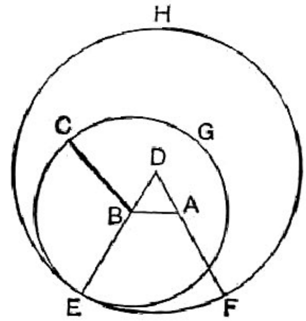
Therefore the triangle **ABC** is equilateral;

and it is described on the given straight line **AB.** Q.E.F.

PROPOSITION 2. PROBLEM.

General Enunciation.

From a given point to draw a straight line equal to a given straight line.



Particular Enunciation.

Let **A** be the given point, and **BC** the given straight line.

It is required to draw from the point **A** a straight line equal to **BC**.

Construction.

Join **AB**,

and on **AB** describe an equilateral triangle **DAB**. I. 1.

From centre **B**, with radius **BC**, describe the circle **CEG**.

Produce **DB** to meet the circle **CEG** at **E**.

From centre **D**, with radius **DE**, describe the circle **EFH**.

Produce **DA** to meet the circle **EFH** at **F**.

Then **AF** shall be equal to **BC**.

Proof. Because **B** is the centre of the circle **CEG**,

therefore **BC** = **BE**,

and because **D** is the centre of the circle **EFH**,

therefore **DF** = **DE**,

and **DA**, **DB**, parts of them, are equal.

Therefore the remainder **AF** = the remainder **BE**. Ax. 3.

And it has been shown that **BC** is equal to **BE**;

Therefore **AF** and **BC** are each equal to **BE**.

But things which are equal to the same thing are equal to one another. Ax. 1.

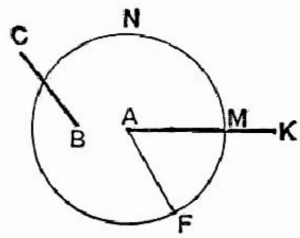
Therefore **AF** = **BC**,

and it has been drawn from the given point **A**. Q.E.F.

PROPOSITION 3. PROBLEM.

General Enunciation.

From the greater of two given straight lines, to cut off a part equal to the less.



Particular Enunciation.

Let **AK** and **BC** be the two given straight lines, of which **AK** is the greater.

It is required to cut off from **AK** a part equal to **BC**.

Construction.

From the point **A** draw the straight line **AF** equal to **BC**, I. 2. and from centre **A**, with radius **AF**, describe the circle **FMN** meeting **AK** at **M**.

Then **AM** shall be equal to **BC**.

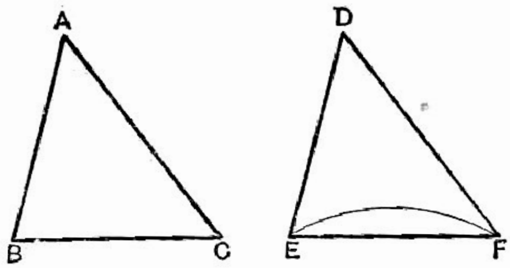
Proof.

Because **A** is the centre of the circle **FMN**,
therefore **AM** = **AF**.
But **BC** = **AF**,
therefore **AM** and **BC** are each equal to **AF**,
therefore **AM** = **BC**, Ax. 1.
and it has been cut off from the given straight line **AK**. Q.E.F.

PROPOSITION 4. THEOREM.

General Enunciation.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal, then shall their bases or third sides be equal, and the triangles shall be equal in area, and their remaining angles shall be equal, each to each, namely, those to which the equal sides are opposite: that is to say, the triangles shall be equal in all respects.



Particular Enunciation.

Let **ABC**, **DEF** be two triangles, which have
the side **AB** = the side **DE**,
and the side **AC** = the side **DF**,
and the contained angle **BAC** = the contained angle **EDF**.

Then shall the base **BC** be equal to the base **EF**, and the triangle **ABC** shall be equal to the triangle **DEF** in area; and the remaining angles shall be equal, each to each, to which the equal sides are opposite, namely,
the angle **ABC** to the angle **DEF**,
and the angle **ACB** to the angle **DFE**.

Proof.

For if the triangle **ABC** be applied to the triangle **DEF**,
so that the point **A** may be on the point **D**,
and the straight line **AB** along the straight line **DE**,
then because **AB** is equal to **DE**, Hyp.
therefore the point **B** must coincide with the point **E**.

And because **AB** falls along **DE**,
and the angle **BAC** is equal to the angle **EDF**, Hyp.
therefore **AC** must fall along **DF**.

And because **AC** is equal to **DF**,
therefore the point **C** must coincide with the point **F**.

Then **B** coinciding with **E**, and **C** with **F**,
the base **BC** must coincide with the base **EF**;
for if not, two straight lines would enclose a space; which is impossible. Ax. 10.

Thus the base **BC** coincides with the base **EF**, and is therefore equal to it. Ax. 8.

And the triangle **ABC** coincides with the triangle **DEF**, and is therefore equal to it in area. Ax. 8.

And the remaining angles of the one coincide with the remaining angles of the other, and are therefore equal to them, namely,

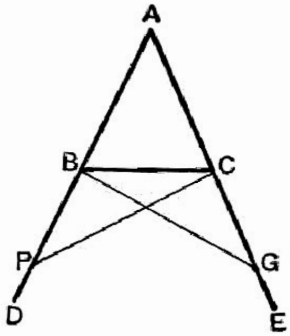
the angle **ABC** to the angle **DEF**,
and the angle **ACB** to the angle **DFE**.

That is, the triangles are equal in all respects. Q.E.D.

PROPOSITION 5. THEOREM.

General Enunciation.

The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced, the angles on the other side of the base shall also be equal to one another.



Particular Enunciation.

Let **ABC** be an isosceles triangle, having the side **AB** equal to the side **AC**, and let the straight lines **AB**, **AC** be produced to **D** and **E** :

then shall the angle **ABC** be equal to the angle **ACB**,
and the angle **CBD** to the angle **BCE**.

Construction.

In **BD** take any point **F** ;
and from **AE** the greater cut off **AG** equal to **AF** the less. I. 3.
Join **FC**, **GB**.

Proof. Then in the triangles **FAC, GAB,**

because
- **FA** = **GA**, Constr.
- **AC** = **AB**, Hyp.
- also the contained angle at **A** is common to both triangles ;

therefore the triangle **FAC** is equal to the triangle **GAB** in all respects ; I. 4.

that is, the base **FC** = the base **GB**,
and the angle **ACF** = the angle **ABG**,
also the angle **AFC** = the angle **AGB**.

Again, because the whole **AF** = the whole **AG**, Constr.
and the part **AB** = the part **AC**, Hyp.
therefore the remainder **BF** = the remainder **CG**. Ax. 3.

Then in the two triangles **BFC, CGB**,

because
- **BF** = **CG**, Proved.
- **FC** = **GB**, Proved.
- also the contained angle **BFC** = the contained angle **CGB**; Proved.

therefore the triangles **BFC, CGB** are equal in all respects;
so that the angle **FBC** = the angle **GCB**,
and the angle **BCF** = the angle **CBG**, I. 4.

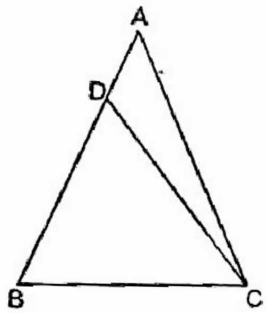
Now it has been shown that the whole angle **ABG** is equal to whole angle **ACF**,
and that the parts of these, namely, the angles **CBG, BCF** are also equal;
therefore the remaining angle **ABC** is equal to the remaining angle **ACB**;
and these are the angles at the base of the triangle **ABC**.

Also it has been shown that the angle **FBC** is equal to the angle **GCB**;
and these are the angles on the other side of the base. Q.E.D.

PROPOSITION 6. THEOREM.

General Enunciation.

If two angles of a triangle be equal to one another, then the sides also which subtend, or are opposite to, the equal angles, shall be equal to one another.



Particular Enunciation.

Let **ABC** be a triangle, having the angle **ABC** equal to the angle **ACB** :

then shall the side **AC** be equal to the side **AB.**

Construction.

For if **AC** be not equal to **AB**,
one of them must be greater than the other.
If possible, let **AB** be the greater ;
and from it cut off **BD** equal to **AC**. I. 3.
Join **DC.**

Proof. Then in the triangles **DBC, ACB,**

because
- **DB** = **AC**, Constr.
- **BC** is common to both,
- and the contained angle **DBC** = the contained angle **ACB** ; Hyp.

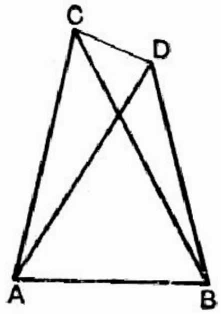
therefore the triangle **DBC** is equal in area to the triangle **ACB**, I. 4.

the part equal to the whole ; which is absurd. Ax. 9.
Therefore **AB** is not unequal to **AC** ;
that is, **AB** is equal to **AC**. Q.E.D.

PROPOSITION 7. THEOREM.

General Enunciation.

On the same base, and on the same side of it, there cannot be two triangles having their sides which are terminated at one extremity of the base equal to one another, and likewise those which are terminated at the other extremity equal to one another.



Particular Enunciation.

If it be possible, on the same base **AB**, and on the same side of it, let there be two triangles **ACB**, **ADB**, having their sides **AC**, **AD**, which are terminated at **A**, equal to one another, and likewise their sides **BC**, **BD**, which are terminated at **B**, equal to one another.

CASE I. When the vertex of each triangle is without the other triangle.

Construction. Join **CD**.

Proof. Then in the triangle **ACD**,

because **AC** = **AD**, Hyp.

therefore the angle **ACD** = the angle **ADC**. I. 5.

But the whole angle **ACD** is greater than its part, the angle **BCD**,

therefore also the angle **ADC** is greater than the angle **BCD**;

still more then is the angle **BDC** greater than the angle **BCD**.

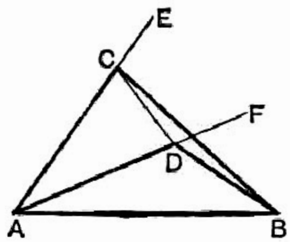
Again, in the triangle **BCD**,

because **BC** = **BD**, Hyp.

therefore the angle **BDC** = the angle **BCD**: I. 5.

but it was shown to be greater; which is impossible.

CASE II. When one of the vertices, as **D**, is within the other riangle **ACB**.



Construction. As before, join **CD** ;
and produce **AC**, **AD** to **E** and **F**.

Proof. Then in the triangle **ACD**,
because **AC** = **AD**, Hyp.
therefore the angles **ECD**, **FDC**, on the other side of the base, are equal to one another. I. 5.

But the angle **ECD** is greater than its part, the angle **BCD** ;
therefore the angle **FDC** is also greater than the angle **BCD** :
still more then is the angle **BDC** greater than the angle **BCD**.

Again, in the triangle **BCD**,
because **BC** = **BD**, Hyp.
therefore the angle **BDC** = the angle **BCD** : I. 5.
but it has been shown to be greater ; which is impossible.

The case in which the vertex of one triangle is on a side of the other needs no demonstration.

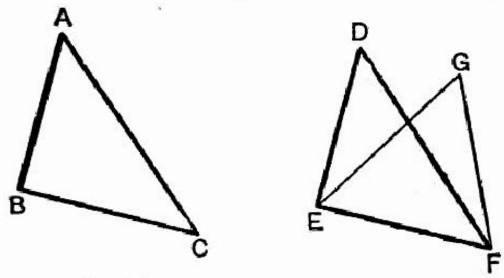
Therefore **AC** cannot be equal to **AD**, and *at the same time* **BC** equal to **BD** Q.E.D.

NOTE. The sides AC, AD are called *conterminous* sides : similarly the sides BC, BD are conterminous sides.

PROPOSITION 8. THEOREM.

General Enunciation.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, then the angle which is contained by the two sides of the one shall be equal to the angle which is contained by the two sides of the other.



Particular Enunciation.

Let **ABC**, **DEF** be two triangles having the two sides **BA**, **AC** equal to the two sides **ED**, **DF**, each to each, namely, **BA** to **ED**, and **AC** to **DF**, and also the base **BC** equal to the base **EF** :

then shall the angle **BAC** be equal to the angle **EDF**.

Proof.

For if the triangle **ABC** be applied to the triangle **DEF**, so that the point **B** may be on **E**, and the straight line **BC** along **EF** ;

then because **BC**=**EF**, Hyp.

therefore the point **C** must coincide with the point **F**.

Then **BC** coinciding with **EF**,

it follows that **BA** and **AC** must coincide with **ED** and **DF** :

for, if not, they would have a different situation, as **EG**, **GF** : then, on the same base and on the same side of it there would be two triangles having their *conterminous* sides equal.

But this is impossible. I. 7.

Therefore the sides **BA**, **AC** coincide with the sides **ED**, **DF**.

That is, the angle **BAC** coincides with the angle **EDF**, and is therefore equal to it. Ax. 8.

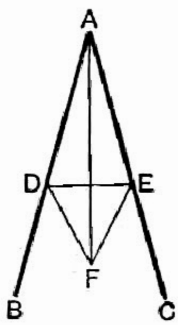
Q.E.D.

Corollary. *If in two triangles the three sides of the one are equal to the three sides of the other, each to each, then the triangles are equal in all respects.*

PROPOSITION 9. PROBLEM.

General Enunciation.

To bisect a given angle, that is, to divide it into two equal parts.



Particular Enunciation.

Let **BAC** be the given angle ;
it is required to bisect it.

Construction. In **AB** take any point **D** ;
and from **AC** cut off **AE** equal to **AD**. I. 3.
Join **DE**.
and on **DE**, on the side remote from **A**, describe an equilateral triangle **DEF**. I. 1.
Join **AF**.

Then shall the straight line **AF** bisect the angle **BAC**.

Proof. For in the two triangles **DAF**, **EAF**,

because { **DA**=**EA**, Constr.
and **AF** is common to both ;
and **DF**=**EF** ; Def. 19.

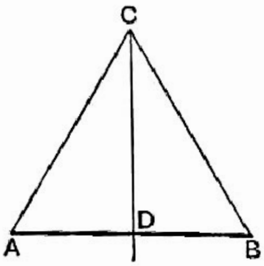
therefore the angle **DAF**=the angle **EAF**. I. 8.

Therefore the given angle **BAC** is bisected by the straight line **AF**. Q.E.F.

PROPOSITION 10. PROBLEM.

General Enunciation.

To bisect a given finite straight line, that is, to divide it into two equal parts.



Particular Enunciation.

Let **AB** be the given straight line :
it is required to divide it into two equal parts.

Construction.

On **AB** describe an equilateral triangle **ABC**, I. 1.
and bisect the angle **ACB** by the straight line **CD**, I. 9.
and let **CD** cut **AB** in **D**.

Proof.

For in the triangles **ACD, BCD**,

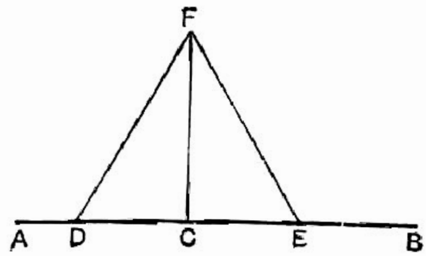
because {
AC = **BC**, Def. 19.
and **CD** is common to both ;
also the contained angle **ACD** = the contained angle **BCD** ; Constr.
}

therefore the triangles are equal in all respects :
so that the base **AD** = the base **BD**. I. 4.
Therefore the straight line **AB** is bisected at the point **D**.
Q.E.F.

PROPOSITION 11. PROBLEM.

General Enunciation.

To draw a straight line at right angles to a given finite straight line from a given point in the same.



Particular Enunciation.

Let **AB** be the given straight line and **C** the given point in it. It is required to draw from the point **C** a straight line at right angles to **AB**.

Construction. In **AC** take any point **D**,

and from **CB** cut off **CE** equal to **CD**. I. 3.

On **DE** describe the equilateral triangle **DFE**. I. 1.

Join **CF**.

Then shall the straight line **CF** be at right angles to **AB**.

Proof. For in the triangles **DCF**, **ECF**,

because $\left\{\begin{array}{l} \mathbf{DC}=\mathbf{EC}, \\ \text{and } \mathbf{CF} \text{ is common to both,} \\ \text{and } \mathbf{DF}=\mathbf{EF}; \end{array}\right.$ Constr. / Def. 19.

therefore the angle **DCF** = the angle **ECF**, I. 8.

and these are adjacent angles.

But when a straight line, standing on another straight line, makes the adjacent angles equal to one another, each of these angles is called a right angle ; Def. 7.

therefore each of the angles **DCF**, **ECF** is a right angle.

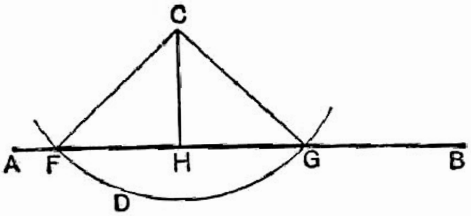
Therefore **CF** is at right angles to **AB**,

and has been drawn from a point **C** in it. Q.E.F.

PROPOSITION 12. PROBLEM.

General Enunciation.

To draw a straight line perpendicular to a given straight line of unlimited length from a given point without it.



Particular Enunciation.

Let **AB** be the given straight line which may be produced in either direction, and let **C** be the given point without it.

It is required to draw from the point **C** a straight line perpendicular to **AB.**

Construction.

On the side of **AB** remote from **C** take any point **D** ; and from centre **C,** with radius **CD,** describe the circle **FDG,** meeting **AB** at **F** and **G.**

Bisect **FG** at **H** ; I. 10.

and join **CH.**

Then shall the straight line **CH** be perpendicular to **AB.**

Join **CF** and **CG.**

Proof. Then in the triangles **FHC, GHC,**

because $\begin{cases} \textbf{FH}=\textbf{GH}, & \text{Constr.} \\ \text{and } \textbf{HC} \text{ is common to both ;} & \\ \text{and } \textbf{CF}=\textbf{CG}, \text{ being radii of the circle } \textbf{FDG} \text{ ;} & \end{cases}$

therefore the angle **CHF**=the angle **CHG** ; I. 8.

and these are adjacent angles.

But when a straight line, standing on another straight line, makes the adjacent angles equal to one another, each of these angles is called a right angle, and the straight line which stands on the other is called a perpendicular to it.

Therefore **CH** is a perpendicular drawn to the given straight line **AB** from the given point **C** without it. Q.E.F.

EXPLANATION OF FIGURES.

Figure 1.

ABCD is a square.

K, L, M, N are the middle points of the sides of the square.

A circle with centre A is drawn, cutting AB in E and AD in F.

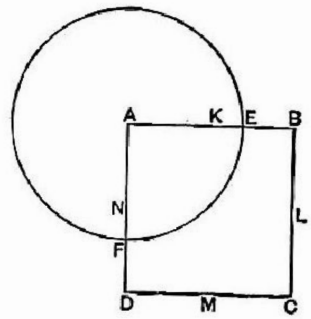


Figure 2.

AB is a diameter of the circle.

OC is a radius, perpendicular to OB.

OD is any other radius.

OE is the bisector of the angle DOB.

AOCF is a square.

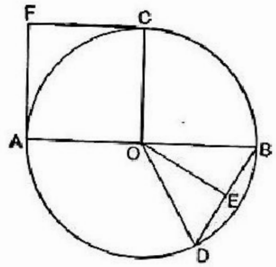


Figure 3.

ABCD, AEFG are two squares with common angle at A.

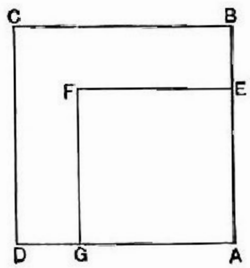


FIGURE 4.

Two circles are drawn with the same centre O (*concentric* circles).

SQOBD, RPOAC are two straight lines through O.

NOTE. It is given that the angle QOP is equal to the angle AOB. (This is proved in Euclid, I., 15.)

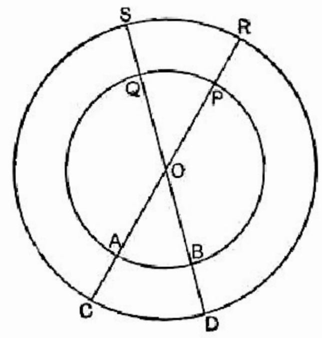
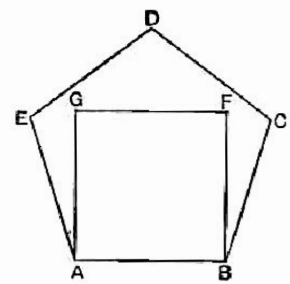


FIGURE 5.

ABCDE is a *regular* pentagon (*i.e.* all its sides and angles are equal).

ABFG is a square.



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