MATHEMATICAL PERSPECTIVE.

By TELFORD PETRIE, D.SC.

"Mathematics depend upon the teacher rather than upon the text-book, and few subjects are worse taught: chiefly because teachers have seldom time to give the inspiring ideas, what Coleridge calls the 'Captain' ideas, which should quicken imagination."—Charlotte M. Mason in An Essay towards a Philosophy of Education, 1925, *p*. 233.

"How interesting arithmetic and geometry might be if we gave a short history of their principal theorems, if the child were meant to be present at the labours of a Pythagoras, a Plato, a Euclid, or in modern times, of a Descartes, a Pascal, or a Leibnitz. Great theories, instead of being lifeless and anonymous abstractions, would become living human truths each with its own history, like a statue by Michael Angelo, or like a painting by Raphael."—M. Fouillée in Education from a Natural Standpoint—quoted on p. 110 of above book.

Miss Mason, in so much that she said and wrote, had the glorious faculty of stimulating the imagination, of supplying food for thought, by a twist of a sentence here, by an illustration there. She certainly practised what she preached; her principles of education were so clearly set out, so understandable by others, chiefly because she used these very principles in her exposition. Her mind and her principles were one. We see, in the first quotation at the head of this article, her feelings about the present-day methods of Educational Mathematics, the twist of the sentence which illuminates what was at the back of her mind, and in the second quotation (though it occurs in an earlier portion of the book) the illustration of what she herself described as "an application of Coleridge's 'captain idea' of every train of thought."

It is not proposed to discuss here the teaching of elementary mathematics as carried out in various ways at the present time, more especially as the subject was so ably dealt with by Miss Gardner at the Children's Gathering in Canterbury in 1925, but the sentences quoted, and others in the same book, have stimulated various thoughts on the more general aspects of the question which it might be of interest to consider.

[p 669]

First of all, there seems to be a very general impression that the acquirement of proficiency in mathematics involves drudgery; not hard work, which harms no one, but heavy all-engrossing labour of a mechanical kind. No doubt there are a number of children with a natural aptitude for mathematics, just as there are some more musically inclined than others, and they, to a large extent, escape this drudgery, because the subject interests them for its own sake without the stimulus of any "captain" ideas. These are usually considered to be in a minority and it is too often assumed that the remainder can only acquire the necessary knowledge by a species of pack-drill, soul-shattering to themselves as it should be distasteful to the teacher. If this is really so, and the writer sincerely hopes that he is exaggerating, then there must be something wrong, not only with the text-books but also with the teacher.

This may be due, in part, to a possible misconception of the uses and purposes of the subject. Its ramifications are so very wide and its uses so various that it is not surprising that some misconceptions arise. Fortunately there exists a book, written by a master-mind, which is an invaluable guide to any teacher on this vital point. It is called *An Introduction to*

Mathematics, and is by Dr. A. N. Whitehead, who was for so many years a Professor of Mathematics in the University of London. It is very accessible, as it is published in the Home University Library series. At the very commencement of his subject Dr. Whitehead points out that "the reason for this failure of the science (i.e., of mathematics) to live up to its reputation is that its fundamental ideas are not explained to the student disentangled from the technical procedure which has been invented to facilitate their exact presentation in particular instances. Accordingly, the unfortunate learner finds himself struggling to acquire a knowledge of a mass of details which are not illuminated by any general conception." He then goes on to discuss, not the teaching of mathematics, but "why it is necessarily the foundation of exact thought as applied to natural phenomenon." Practically the whole of the subject required up to University standard is then analysed from this point of view.

Now the subject matter of mathematics, considered as a whole, is so vast that, as we have seen, it ranks as one of the sciences. But it is far more than that; it not only forms a
[p 670]
very considerable science of its own (pure mathematics) but it acts as a hand-maiden to all the other sciences (applied mathematics). And this is the first distinction which is often lost sight of when the subject is first taught. Your child with a natural bent for mathematics belongs to the ranks from which the pure mathematicians are drawn, but there is a far larger army of children without any predilection for the subject who could be brought to enjoy and appreciate mathematics as an applied science if they have the subject presented to them in this light from the beginning. As far as the writer is aware, no such an attempt is made during the school age, and every pupil is asked to treat the subject for its own sake. Against this statement may be set the fact that our junior arithmetics and algebras (though not so much geometries) teem with examples of applications; from the number of apples that Bobbie can buy for a shilling, via the papering of rooms to vexed problems of stocks and shares and income tax. Granted, they do. But how many solve the problems in the way in which they are actually solved by the shopkeeper, the paper-hanger, or the stockbroker and banker? For the most part numerical questions in present-day text-books are cast into such forms from the misguided attempt to make the subject matter interesting. Very seldom is the problem solved as it actually would be in real practice, and therefore the mathematics are not, in the correct sense, applied. This must not be taken as a plea that the practical methods of solution should be used instead—the elaborate interest-tables of the banker or the addition machines of commerce, for instance,—but rather that the examples themselves should be exchanged for those which are actually solved by applied mathematics—the hundred and one, of all grades of difficulty, which could be drawn from physics, chemistry, engineering and a host of other sciences.

But the discussion is getting on a little faster than was intended. The difficulties and their solution lie a good deal deeper than the confusion between pure and misapplied mathematics. As Dr. Whitehead points out (p. 8) "technical facility is a first requisite for valuable mental activity: we shall fail to appreciate the rhythm of Milton or the passion of Shelley, so long as we find it necessary to spell the words and are not quite certain of the forms of the individual letters." It would seem that this sentence contains the main reason
[p 671]
why elementary mathematics so often becomes a drudgery. The student is never brought to realise that what he is doing, when he starts, is not arithmetic or algebra or geometry, but merely the language and symbolism of the subject. Just as he has to learn the notes and staves of music before he can even start to play the piano so must he learn addition and

subtraction, multiplication and division before he can start arithmetic, or the symbols and conventions before he can start algebra, or the axioms and first propositions before he commences geometry. Nor does the distinction end here. After the notes of music have been learnt we have to go on to other matters; clefs and keys, counterpoint and harmony, all of which are part of the language of music. In just the same way the language of mathematics has to be added to as new conceptions arise—and the latest one of all, geodesics, is so complicated and revolutionary that very few mathematicians, even in the highest ranks, have yet mastered it. But Einstein found its invention necessary before he could solve his general theory of Relativity.

Why do we have all this language of mathematics? Surely, to leave our minds free for the real problems of the subject and not to burden them with the spelling and pronunciation of the separate words. One cannot help quoting Dr. Whitehead again (p. 61): "It is a profoundly erroneous truism, repeated by all copy books and by eminent people when they are making speeches, that we should cultivate the habit of thinking of what we are doing. The precise opposite is the case. Civilization advances by extending the number of important operations which we can perform without thinking about them. Operations of thought are like cavalry charges in battle—they are strictly limited in number, they require fresh horses, and must only be made at decisive moments." There is no subject in our syllabus that illustrates this paradox better than mathematics.

The illustration may be carried even further. Not only have we to learn the language of mathematics before we can solve mathematical problems, but we can use both the language and the solution of the mathematical problem to clarify and explain other branches of knowledge. In other words we can look upon the whole subject of mathematics as a tool, an instrument, or as an "operation which we can perform without thinking about it." This is the sense in which a knowledge of

[p 672]

mathematics is most useful to the average individual, but it is open to doubt whether this is the way in which the subject is presented to the average pupil of to-day. Certainly the examination papers which he is expected to pass before he enters the University show little signs of it, and consequently the text-books, designed in the main for some such examinations, do not draw the clear distinctions which they should between pure and applied mathematics. The average student is bewildered, lost in the mass of detail and has no definite idea of where he is supposed to be going. Can one blame him if he feels the work to be a drudgery? It always will be so as long as the means to an end and the end itself are mixed up in confusion.

To return now to the "captain" ideas of Miss Mason and M. Fouillée, nothing could be more delightful than to turn the mathematical periods on the time-table, or some of them, into accounts of the lives and labours of the great pioneers. But there would seem to be very great difficulties in the way. To begin with, very few clear and simple narratives of such pioneers exist. Books like Sir Oliver Lodge's *Pioneers of Science*, which deals with astronomical mathematicians, and Fournier's *Wonders of Physical Science*, are certainly exceptions, in that they can be understood by anyone without more than a very elementary knowledge of mathematics; but they do not nearly cover the necessary ground, and the majority of such works are much too technical to be suitable for the beginner. But there is another serious objection, even if the right kind of book was multiplied; and that is the question of time. Mathematics, fascinating as it is to the expert, must take their place in the syllabus with many other subjects, and the time available in most cases is barely sufficient to cover the grammar of the subject, let alone any knowledge of the grammarians. The best training for proficiency in mathematics is practice by the pupil himself. There is no short cut in this respect and the more work a student does in this connection the more efficient he is bound to become. In the Parents' Union School homework is not allowed. All the more reason, therefore, why the times for arithmetic, algebra and geometry should be utilised to the full by students working out examples and applying their knowledge of the rules for themselves.

The work can, nevertheless, be made inspiring if the pupil is allowed to realise why he is doing any particular thing, if a clear distinction is kept between the language of mathematics

[p 673]

and its application and if he is shown, as soon as possible, some results of such applications. For this reason the writer would advocate the teaching of the use of the slide rule and of logarithms in arithmetic immediately after addition, subtraction, multiplication and division have been learnt. The theory of these aids to calculation may safely be left till a good deal later, but elementary decimals could be taught at the same time as the first four rules. They will give the pupil a sense of balance and proportion which is so necessary to the proper understanding of the subject and will greatly help the proper use of both logarithms and the slide-rule. At present their use is practically prohibited on the mistaken idea that they would make mathematics too easy! But that is the whole object of the symbolism of mathematics, as witness the words of Dr. Whitehead already quoted, and the writer will never forget the way in which the theory of logarithms fascinated him long after he had been familiar with their use. There is another important reason why such aids to calculation should be taught much earlier on than is frequently the case and that is the great advantage they have of providing a check on one's working. The essence of all numerical solutions is that they should be correct, and the best way of ensuring this is to work out the problem in more ways than one. The processes of addition and subtraction are simpler and quicker than long multiplication and division, and there is less likelihood of a numerical slip being made. Logarithms simply provide a means of reducing all multiplication and division to addition and subtraction, and moreover enable any number of figures to be dealt with at the same time. The slide-rule consists of two similar logarithmic scales so that these processes of addition and subtraction can be done mechanically, and as the divisions are marked in their corresponding natural numbers and not as logarithms the answer can be read off directly to three significant figures. Logarithms as a rule are used to give four correct figures in the answer though they can be obtained up to twenty significant figures if necessary! This adds another valuable point to their use. Judgment is required as to how great an accuracy the problem requires. This gives an early appreciation of the sense of proportion. Slide-rules, for instance, are quite accurate for all percentage problems, whilst logarithms should be used for all solutions which have to be accurate to one part in a thousand. It is surprising how very

[p 674]

seldom practical problems require a greater accuracy than that. It must never be forgotten, though one fears that it often is, that it is actually telling a lie to give an answer to a greater degree of accuracy than the data warrant. This applies, in particular, to all numerical measurements obtained by experiment. Every instrument used to record those measurements is liable to an error, it may be one in a thousand or it may be, and very often is, one in a hundred. In the latter case, numerical results based on such measurements have no real significance beyond three figures and even logarithmic calculations are too accurate if the final figure is left in. Again there is a very common method of safeguarding reliability by taking a large number of readings, adding them all up and dividing by the number of readings to get an average result. Here again this average result should not be worked out to a greater degree of accuracy than the original measurements warrant. It is true that there are processes by which a greater degree of accuracy may be deduced, but these belong to the region of higher mathematics, and need not be elaborated here.

To turn now to other forms of elementary mathematics which are, or are not, of particular use in the applied sense. In arithmetic the evaluation of square roots and cube roots is important, but here again the use of logarithms is much more convenient than the long-hand methods usually taught. This, however, should come later, as the process involved (of negative logarithms) is not quite so simple or so easy to manipulate as simple multiplication and division. Fractions are useful in this country because of our dear old systems of weights, measures and coinage, but otherwise they are safer and more easily manipulated as decimals. Therefore the conversion of fractions to decimals and *vice versa* should be stressed.

Prime factors, highest common factor and lowest common multiple, on the other hand, belong to the science of pure mathematics and have little, if any, practical use. The list could be extended, but enough has been given to illustrate the point. Algebra, being an abstract science, is frequently found to be more difficult to understand at first; and the task is not lessened by the new and more complicated symbolism which has to be acquired. Its value, however, is so great in applied mathematics that it can very soon be made interesting by the innumerable illustrations available. Simple, simultaneous and quadratic equations form the mainstay of this work, whilst
[p 675]
in certain sciences, such as thermodynamics, a clear knowledge of surds and indices is invaluable. The beauty of this branch of mathematics is very often enhanced by the use of graphs drawn on squared paper showing how our old, though at first rather obscure, friends, x and y, interact on one another. There again the graphical solution, though often not so accurate, forms a most convenient check on analytical reasoning, and frequently comes to the rescue by providing the only method by which a solution can be obtained. Now practically any child can be taught to draw a graph as easily as to use a slide-rule, but it is to be feared that here again the method is not encouraged as it might be.

Lastly, take geometry, the oldest branch of the subject, for without a knowledge of it the Pyramids could not have been designed. No part of elementary mathematics has been approached in more different ways, from the teaching of pure Euclid to the graphical methods which permit all kinds of drawing instruments to be utilised. Unlike arithmetic and algebra, the difference between pure and applied geometry is quite distinct and need not be confused at all, even in its most elementary stages. The subject can be made quite interesting for its own sake and should appeal to every child as soon as he or she can appreciate a detective story.

The most advanced branches of the science in its pre-University stage, trigonometry and the elementary calculus, need not be discussed beyond pointing out that the method of presenting them advocated here is even more applicable in their case than in the more elementary branches of the subject. With this reservation, however, that their application is more specialised, and many folk manage to become excellent citizens without any knowledge of them whatsoever.

It is to be feared that some of the suggestions put forward in this paper are outside the realms of present-day school politics, but the essence of the mathematical game is proficiency in manipulation of its language; and that could undoubtedly be made more interesting if all legitimate aids were invoked at the earliest possible moment and then utilised in the solution of real problems. The criterion in the choice of such problems should be that the answer itself is worth obtaining and will interest the pupil. A sense of balance and of proportion should be cultivated from the very start, so that the child will set out with a clear idea of mathematical perspective.